Eight-Book Mathematics Bundle

(The family math book, something for everyone in the family)

Topic by topic, the most-user-friendly and most useful math books in the world

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(2,738 pages; 447 Lessons)

This ebook has something for everyone in the family, or at school; and covers all the basics of all the important math topics from elementary school, through high school to college. Every topic is well-covered. Some of the books are also ideal for developmental or remedial math programs, as well as adult education and distance learning programs at two-year (US) colleges; and much more, these books are ideal for final exams review.You may click or scroll from page to page, from chapter to chapter, and from book to book.

A. A. FREMPONG

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19×19=361	20×19=380	21×19=399	22×19=418	23×19=437	24×19=456
19×20=380	20×20=400	21×20=420	22×20=440	23×20=460	24×20=480

If you can remember a needed information, you make decisions faster, you learn faster, you work faster, and you are more productive.

Yes, you can memorize them.

Squares of Natural Numbers

$1\times1=1$
$2\times2=4$
$3\times3=9$
$4\times4=16$
$5\times5=25$
$6\times6=36$
$7\times7=49$
$8\times8=64$
$9\times9=81$
$10\times10=100$
$11\times11=121$
$12\times12=144$
$13\times13=169$
$14\times14=196$
$15\times15=225$
$16\times16=256$
$17\times17=289$
$18\times18=324$
$19\times19=361$
$20\times20=400$
$21\times21=441$
$22\times22=484$
$23\times23=529$
$24\times24=576$
$25\times25=625$
$26\times26=676$
$27\times27=729$
$28\times28=784$
$29\times29=841$
$30\times30=900$
$31\times31=961$
$32\times32=1024$
$33\times33=1089$
$34\times34=1156$
$35\times35=1225$
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$43\times43=1849$
$44\times44=1936$
$45\times45=2025$
$46\times46=2116$
$47\times47=2209$
$48\times48=2304$
$49\times49=2401$
$50\times50=2500$

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In Memory of My Parents

Mom:
She was a devoted mother, sharing, kind, kinder to strangers and generous to a fault. She never cursed, she never hated; she never cheated, and she never envied. She never lied, and she never got angry. Once, she nursed an almost dying stranger renting a room in her house back to good health to the extent that the relatives of this renter later travelled one hundred miles just to thank mom. She was always peaceloving and forever forgiving. An angel once lived on this earth to serve others.

Dad:
A great dad, kind, generous and forgiving. He emphasized and was an example of both formal education and self-education. A veterinarian, a bacteriologist, an Associate of the Institute of Medical Laboratory Technology (UK), a Fellow of the Royal Society of Health (UK); an incorruptible civil servant; his book on ticks has always inspired me to write whenever the need arises.

PREFACE

This Eight-Book Math Bundle has something for everyone in the family; and covers all the basics of all the important math topics from elementary school, through high school, to first year college.

Some of the books are also ideal for developmental or remedial math programs, as well as adult education and distance learning programs at two-year (US) colleges. .

Since the various topics are accessible with the click of the mouse, these books are ideal for comprehensive studying and learning and review, as well as for short-term programs such as mini-sessions and immersion programs.

The materials in these books are based on materials from the paper editions which have been used as textbooks.

If you are interested also in the paper versions, visit the companion website, **Yellowtextbooks.com**

From time to time, the student is encouraged to visit the **Microtextbooks.com** website to view the improvements in the content of the books

Navigation in the Ebook

This book is one of the most user-friendly ebooks both in content and in navigation through the book. You can use the links on the pages alone; or the contents-bookmarks links; or a combination of the page links (which also contain the Table of Contents) and the contents-bookmarks links. The first approach emulates how you use paper books. To go to the Lessons directly, use the content-bookmarks links. Also, be familiar with the "Previous View", "Go back", or "Go Back " navigation commands, since you would like to return to the previous view; otherwise, you would be lost and lose continuity in reading.

A. A. Frempong
New York.

NOTE TO THE STUDENT

About being a student

Definition: A student is one who studies (a lot). If one attends classes and does not study, one is not a student, but a class attendee. If one does not attend classes but studies a lot, one is a student.
All learning is self-teaching. Others can only help a student to learn. Since you
are going to self-teach, you need user-friendly books. The best teacher is a well-written textbook, and not a human being. If you have a user-friendly book, you have the best teacher for 24 hours a day, 365 days a year, for the rest of your life, and also for future generations of your family. **Every book in this bundle is very user-friendly**, On every topic, the author spent sufficient time to present the material in such a way that students enjoy studying from them. In studying, it is good practice to refer to other books of the same title by other authors. You can search in the library, the bookstores, on-line, or refer to other textbooks and notebooks used by family members.
I hear and read about complaints against teachers and professors. These complaints should be directed towards textbooks user-friendliness. In order to master any topic, you first study it, followed by repetition, repetition and repetition. Without repetition, you will not master any topic; and when it comes to repetition, only you can afford the time for repetition. Even after having mastered a topic, from time to time, you must review the material learned previously. Your teachers or professors are there to help you learn. They cannot and will not have sufficient time to "put knowledge into a student's head". If any of these books is not the course textbook, use it to help you understand and remember the topics covered. After having found these books to be useful, tell your friends and family members about them, and they would be appreciative for telling them.

About studying

Begin to master the definitions and the solutions of the sample problems.
To master a sample problem, study the sample problem and its solution process first, and then try to do the same problem without looking at the solution process. If you are able to repeat the solution process successfully, you have mastered the sample problem. If you can solve similar problems without any reference to this book or any other source, you have also mastered similar problems. For some problems, two or more methods are presented**.** Study the various methods and decide which methods you would like to remember; but always be aware of the existence of the other methods, in case the need arises. After having mastered the sample problems, try the exercise problems at the end of the lesson. You may refer back and forth to the solved problems when you do not remember how to proceed. Finally, **repeat** all of the above from time to time.
It must be emphasized that **repetition** is very important.

Advice for exams and final exams

You should always aim at getting an "A" on every exam, and especially on the final exam, since some instructors weigh the final exam more than a class exam. Show all work even if the exam is of the multiple-choice-answer type. Since the final exam may be given a few days (or the next day) after the last class meeting, do not wait to begin preparing for the final exam after the last class meeting; but rather, you should begin preparing for the final exam, at least, three weeks before the final exam date/ Confer with your instructor to obtain all the topics that would be covered by the end of the semester or term. If you are working part-time or full-time-time, it would be helpful if you are on vacation from work during the last two weeks of the semester. If you are offered a new employment, let the first day on your new job be after the last final exam.

About Using the Index of Each Book

Each book has an index. To use any entry in the index, you must first be sure that the index is for the book being read.

As a reminder, in any book, do not dwell on the few inadvertent errors you may find, but rather concentrate on what is useful to you.

For any book to be useful,, it is **important** to **understand, remember, apply**, and $\boxed{\textbf{remember}}$ the material covered.

Wishing you Good Luck on the exams.
A. A. Frempong

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CHAPTER 7

Arithmetic

Lesson 21: **Percent (%) and Inter-conversions**
Lesson 22: **Calculations Involving Percent**
Lesson 23: **More Applications Involving Percent**

Lesson 21

Percent (%) and Inter-conversions

Some interpretations of the percent symbol "%":

1. **Over hundred**: For example, 20% means $\frac{20}{100}$ (Twenty over hundred or 20 divided by 100)
2. **For each hundred**: For example, a savings account with a 5% interest rate pays the depositor $5 for each $100 in the account. (Five for each hundred)
3. **Hundredths:** For example, 20% means $\frac{20}{100}$ or .20 (Twenty hundredths)
4. **As** a **number** out of **100**: For example, a grade of 80% on a test means a student got 80 points out of 100 points.

Changing a decimal (or any fraction) to percent

Procedure: Multiply by 100%. (i.e., multiply by 100 and attach the percent symbol "%".

To change a decimal to percent, move the decimal point two places to the right and attach the percent symbol. (Note that moving the decimal point two places to the right is equivalent to multiplying by 100)

Example 1 Convert .74 to percent.
Solution .74 = 74% .

Example 2 Convert .008 to percent.
Solution .008 = .8%

Example 3 Convert 1 to percent.
Solution 1 = 100%

Example 4 Convert 12 to percent.
Solution 12. = 1200%

Example 5 Change $\frac{1}{4}$ to percent

Method 1 $\frac{1}{4} \times \frac{100\%}{1} = 25\%$

Method 2 $\frac{1}{4} = .25 = .25 \times 100\% = 25\%$ (Changing to decimal first, and then changing to percent)

Changing a percent to a decimal

Procedure: Divide by 100%. (i.e., drop the percent symbol "%" and divide by 100) or apply the meaning of the percent symbol to change the percent to a decimal fraction and then easily to a decimal.

To change a percent to a decimal, drop the percent symbol and move the decimal point two places to the left . (**Note** that moving the decimal point two places to the left is equivalent to dividing by 100)

Case 1: Example 1 $74\% = .74$ ($74\% = \frac{74}{100} = .74$)

Example 2 $.8\% = .008$ (or $.8\% = \frac{.8}{100} = .008$)

Example 3 $145\% = 1.45$

Example 4 $14.5\% = .145$

Example 5 $.145\% = .00145$

Case 2: Example 6 Convert $84\frac{2}{3}\%$to a decimal

$$84\frac{2}{3}\% = \frac{84\frac{2}{3}}{100}$$

$= .84\frac{2}{3}$ (moving the decimal point two places to the left)

Example 7 Convert $84\frac{2}{3}\%$ to a decimal rounded-off to the nearest **hundredth.**

$84\frac{2}{3}\% = 0.846... \approx 0.85$ $\left(\frac{254}{3}\% = 84.6...\% = .846... \approx .85\right)$

Example 8 Convert $8\frac{1}{4}\%$ to a decimal.

$8\frac{1}{4}\% = 0.08\frac{1}{4}$ <-------- Complex decimal.

However, since $\frac{1}{4} = 0.25$, a terminating decimal,

$8\frac{1}{4}\% = 0.08\frac{1}{4} = 0.0825$ (Also, $8\frac{1}{4}\% = \frac{33}{4}\% = 8.25\% = \frac{8.25}{100} = .0825$.)

Example 9 Convert $4\frac{2}{3}\%$to a decimal

$$4\frac{2}{3}\% = \frac{4\frac{2}{3}}{100}$$

$= .04\frac{2}{3}$ (moving the decimal point two places to the left and writing a zero to hold place)

To check: Let us convert $.04\frac{2}{3}$ to a percent.

$.04\frac{2}{3} = .04\frac{2}{3} \times 100\%$

$= 4\frac{2}{3}\%$ (moving the decimal point two places to the right and attaching the percent symbol)

Changing a Percent to a Fraction (in its lowest terms)

Example 1. Convert 25% to a fraction in its lowest terms.

Solution: $25\% = \frac{25}{100} = \frac{1}{4}$.

Example 2. Convert 23% to a fraction in its lowest terms.

Solution: $23\% = \frac{23}{100}$.

Example 3. Convert 74% to a fraction in its lowest terms

Solution: $74\% = \frac{74}{100} = \frac{37}{50}$.

Example 4 Convert $4\frac{2}{3}\%$ to a fraction

$$4\tfrac{2}{3}\% = \frac{4\frac{2}{3}}{100}$$

$$= \frac{14}{300}$$

$$= \frac{7}{150}$$

Example 5 Convert $.4\frac{2}{3}\%$ to a fraction

$$.4\tfrac{2}{3}\% = \frac{.4\frac{2}{3}}{100}$$

$$= .4\tfrac{2}{3} \div 100$$

$$= \frac{14}{30} \div 100$$

$$= \frac{14}{3000}$$

$$= \frac{7}{1500}$$

Note: Attaching the % symbol is equivalent to dividing by 100; and dropping the % symbol is equivalent to multiplying by 100.

Lesson 21 Exercises

A. Convert to a decimal**: 1.** 23%**; 2.** 8.25%; **3.** 0.8%; **4.** 10%**; 5.** 10.5%**; 6.** 8%

Answers: **1**. 0.23 ; **2**. 0.0825 **3**. 0.008; **4.** 0.10 or 0.1; **5**. 0.105; **6**. 0.08

B. Convert to a decimal**: 1.** $2\frac{2}{3}\%$**; 2.** $25\frac{2}{7}\%$; **3.** $4.3\frac{5}{11}\%$; **4.** $0.16\frac{2}{3}\%$**; 5.** 8.2%**; 6.** $64\frac{3}{11}\%$

Answers: 1. $0.02\frac{2}{3}$**; 2.** $0.25\frac{2}{7}$; **3.** $0.043\frac{5}{11}$; **4.** $0.0016\frac{2}{3}$**. 5.** 0.082; **6.** $0.64\frac{3}{11}$

C. Convert to a fraction in its lowest terms

1. 24% **; 2.** 63% ; **3.** 96%; **4.** 8.2%

Answers: 1. $\frac{6}{25}$**; 2.** $\frac{63}{100}$; **3.** $\frac{24}{25}$; **4.** $\frac{41}{500}$**.**

D. Convert to a fraction in its lowest terms

1. $2\frac{2}{3}\%$**; 2.** $25\frac{2}{7}\%$; **3.** $4.3\frac{5}{11}\%$; **4.** $0.16\frac{2}{3}\%$**.**

Answers: 1. $\frac{2}{75}$**; 2.** $\frac{177}{700}$; **3.** $\frac{239}{5500}$; **4.** $\frac{1}{600}$.

Lesson 22
Calculations Involving Percent (%)

In calculations involving percent, three main quantities are involved, namely the percentage, the base, and the rate percent. Some authors call the percentage the amount.
In these problems, you are usually given two of these quantities and you are asked to find the third quantity.

***Percentage:** This is what is obtained when a percent is taken of a number.

Base: This is the number **of** which a percent is taken.

Rate: This the **percent** that is taken of a number.

There are formulas relating these three quantities:

1. **percentage = base × rate%**
2. **base = percentage ÷ rate %**
3. **rate% =** $\dfrac{\textbf{percentage}}{\textbf{base}} \times \textbf{100\%}$ (i.e. rate% = (the ratio of percentage to base) × 100%)

You do not need to memorize the first formula, provided you note that "**of**" implies multiply.
Memorize the second and the third formulas (even though some of the methods discussed below do not need the recall of these formulas). *Some authors call the percentage the **amoun**t and suggest the proportion

$$\frac{r}{100} = \frac{\text{Percentage}}{\text{base}} = \frac{r}{100} = \frac{A}{B}$$, where r = rate, A = Amount and B = Base.

Finding the Percentage

Example 1 Find 20% of 72 (i.e. **Finding the Percentage**)

(Note: "%" means over 100. Example: 20% = $\frac{20}{100}$

20% of 72 ("of" means multiply)

Step 1: $= \frac{20}{100} \times \frac{72}{1}$ <---------Simplify this by any of the methods discussed below.

Step 2:

Method 1: $\frac{20}{100} \times \frac{72}{1} = .20 \times 72 = 14.40 = 14.4$ (Using decimals)

Method 2: $\frac{20}{100} \times \frac{72}{1} = \frac{20 \times 72}{100} = \frac{1440}{100} = 14.40$ or 14.4 (Multiplying numerators and dividing by 100)

Method 3: $\frac{\overset{1}{\cancel{20}}}{\underset{5}{\cancel{100}}} \times \frac{72}{1} = \frac{72}{5} = 14\frac{2}{5}$ or 14.4 (using cancellation)

The most convenient method will depend on the type of numbers involved, and whether we want the answer as a decimal, as a fraction, or as a mixed number.
For example, if there are common factors in the numerator and the denominator, cancellation, (Method 3) may be more convenient; but if there are no common factors , use Method 1 or Method 2. In any case, it is a good practice to set up the problem as in Step 1. The next example will show the usefulness of setting up the problem before proceeding to simplify.

Example 2 Find $4\frac{2}{7}\%$ of 28000.

Step 1 : Translating: $4\frac{2}{7}\%$ of 28000

$$= \frac{4\frac{2}{7}}{100} \times \frac{28000}{1}$$

Step 2: $= \frac{30}{700} \times \frac{28000}{1}$ <-----------Simplify this by any method.

The easiest method is by cancellation, since there are common factors in both the numerator and the denominator.

$$\frac{30}{{}_1\cancel{700}} \times \frac{\overset{40}{\cancel{28000}}}{1} = 30 \times 40 = 1200$$

Note that steps 1 and 2 are important. Do not round off $4\frac{2}{7}\%$ as .04% because you will not get the exact answer. If the rate % were say, 70%, then you could immediately write 70% = .70 and then similarly write $70\frac{1}{4}\% = .7025$; but better, $70\frac{1}{4}\% = \frac{281}{400}$.

Finding the Base (Finding the Original Number)

Note that the next problem is different from the last two examples both in the wording of the problem and how we solve it. In the last two problems, we multiplied. In the next problem; we will divide, but, we must note which number is the divisor.

Example If 20% of a number is 64, what is the number?

There are a number of methods for solving this problem. We will discuss five methods, which include algebraic methods. You may skip the algebraic methods if you do not have the algebraic background to allow you to follow the procedure. Later, in Chapter 7 (percent problems) we will repeat the algebraic method.

Method 1: Using Formula

Base = Percentage ÷ Rate %

In the above problem, 64 is the percentage. (The percentage is sometimes called the " is number". It is the number that (usually) immediately follows or precedes the word "is " in the word problem.

20% is the rate%

base = percentage ÷ rate %

$$= 64 \div 20\%$$

$$= \frac{64}{1} \div \frac{20}{100}$$ <-----you can simplify this by approach 1 or 2 below.

Approach 1. $\frac{64}{1} \times \frac{\overset{5}{\cancel{100}}}{\cancel{20}} = 64 \times 5 = 320$

Approach 2. 64 ÷ .20 (by long division) $\begin{array}{r} 320 \\ 20\overline{)6400} \end{array}$

The number is 320.

Method 2 Using Algebra

Let the number be x.

Then, "20% of the number is 64" translates to $\frac{20x}{100} = 64$

i.e. $\frac{20x}{100} = 64$

Solve for x: $\frac{\overset{1}{(\cancel{100})}}{\underset{1}{(\cancel{20})}} \, \frac{\overset{1}{\cancel{20}x}}{\cancel{100}} = 64\frac{\overset{5}{(\cancel{100})}}{(\cancel{20})_{1}}$

(or $20x = 100 \times 64$

$x = 320$ (or $x = \frac{100 \times 64}{20} = 320$)

Again, the number is 320 .

Note also that since 20% of the number is 64, the number must be greater than 64.

Method 3 If 20% of the number = 64

then 1% of the number $= \frac{64}{20}$

and 100% of the number $= \frac{64}{20} \times \frac{100}{1}$

$= 320.$

Method 4 "**Ratio Method**" **(Arithmetic)**

(This method follows from method 3.)

If 20% of a number = 64

$$\text{then, } 100\% \text{ of the number } = \frac{64}{1} \times \frac{\overset{5}{\cancel{100\%}}}{\cancel{20\%}}$$

$= 320$

In method 4, we used a very useful principle which states that " If more, the smaller divides, and if less, the larger divides". That is, since we expect 100% of the number to be greater than 64, in forming the fraction involving 20% and 100%, the smaller of 20% and 100% is the divisor (smaller divides), and hence we used the fraction,

$\frac{100\%}{20\%}$ <----- " smaller divides" (We multiplied by an improper fraction)

However, if we had expected the number (answer) to be less than 64, we would have used the fraction

$\frac{20\%}{100\%}$ <----- "larger divides" (i.e., we would have multiplied by a proper fraction)

Method 5 **Using Proportion**

20% is to 64 as 100% is to x, where 100% of the original number (the base) is x.

Translating the proportion,

$\frac{20\%}{64} = \frac{100\%}{x}$ (or $\frac{.20}{64} = \frac{1}{x}$)

Solve for x: $20\%\, x = 100\%\,(64)$ (or $.20x = 64$)

$x = \frac{100\%}{20\%}(64)$ (or $x = \frac{64}{.20}$

$x = 320.$

Note: Methods 3, 4 and 5 are basically the same.

Note that 20% as a fraction is $\frac{20}{100} = \frac{1}{5}$.

Therefore, in the above problem, instead of asking "if 20% of a number is 64, what is the number?", we could have asked "if $\frac{1}{5}$ of a number is 64, what is the number? "

Example If $\frac{1}{5}$ of a number is 64, what is the number? We can solve this by any of the methods discussed above. Let us use the algebraic method.

Solving Algebraically:

Let the number be x.

Then $\frac{x}{5} = \frac{64}{1}$ (Note that $\frac{1}{5}x = \frac{x}{5}$)

$(5)\frac{x}{5} = \frac{64}{1}(5)$ or $x \times 1 = 5 \times 64$ (by cross-multiplication)

$x = 320$ or $x = 320$.

The number is 320.

(See also Method 1 of the preceding example)
Go over the last two problems, and note the differences between how the questions are worded and how they are solved. You **must remembe**r how each problem is worded and how to proceed to solve it.
Note also that the previous example was a direct proportion problem. Direct proportion involves a relationship between two quantities whereby as one quantity **increases** the other quantity also i**ncreases.** See p.64 for more examples on direct proportion and also examples on inverse (indirect) proportion whereby as one quantity **increases** the other quantity **decreases** and vice versa.

Example Mary spends 20% of her weekly income on food. If she spends $64 on food every week, what is her weekly income?

Solution : We could reword this problem in the familiar form as " If 20% of a number is 64, what is the number?"; and then use exactly the same method as in the example, above.

Answer: $320 (numerically, the same answer as for the last example).

Finding the Rate %

Example 1 What rate % of 24 is 15?

Method 1

$$\text{By formula: Rate\%} = \frac{\text{Percentage}}{\text{Base}} \times 100\%$$

In this problem , the percentage is 15. The percentage is sometimes referred to as the " is number". It is the number that follows or precedes the word "is" if the problem is worded in the above form. The base is 24. The base is sometimes referred to as the "of number". It is the number that (usually) follows the word "of" if the problem is worded in the above form.

$$\begin{aligned} \text{Then, rate \%} &= \frac{15}{24} \times \frac{100\%}{1} \\ &= \frac{{}^{5}\cancel{15}}{\cancel{24}_{8}} \times \frac{100\%}{1} \\ &= .625 \times 100\% \\ &= 62.5\% \text{ or } 62\tfrac{1}{2}\% \end{aligned}$$

Method 2

Step 1: Form a fraction using the mnemonic device " $\frac{\text{is number}}{\text{of number}}$ "

Then , we obtain $\frac{15}{24}$

Step 2: Change $\frac{15}{24}$ to percent by multiplying by 100 and attaching the '"%" symbol

$$\frac{{}^{5}\cancel{15}}{\cancel{24}_{8}} \times \frac{100\%}{1} = \frac{500\%}{8}$$

$$= 62\tfrac{1}{2}\% \text{ or } 62.5\%$$

Method 3 Using algebra <--You may skip this method if you do not have the algebraic background to allow you to follow the procedure. Later, in Chapter 7, we will repeat this method.

Let $x\%$ be the rate %. We will write an equation in terms of x, and solve for x.
From the wording, "$x\%$ of 24 is 15" translates to:

$$\frac{x}{100} \times \frac{24}{1} = 15$$

$$\frac{24x}{100} = \frac{15}{1}$$

$24x = 100 \times 15$ (by cross-multiplication or by multiplying both sides of the equation by 100)

$x = \frac{100 \times 15}{24}$ (by dividing both sides of the equation by 24)

$x = 62\frac{1}{2}$

$\therefore$ the required rate = $62\tfrac{1}{2}\%$.

Example 2 What rate % of 18 is 44?

Method 1 base = 18 (the "of" number)
percentage = 44 (the "is" number)

$$\text{rate } \% = \frac{\text{percentage}}{\text{base}} \times 100\%$$

$$= \frac{\overset{22}{\cancel{44}}}{\underset{9}{\cancel{18}}} \times \frac{100\ \%}{1}$$

$\therefore$ the required rate is $244\frac{4}{9}\%$

Note in the above problem that, the larger number does NOT have to be in the denominator. The base (the "of" number) must always be in the denominator. Therefore, in forming the fraction, ignore the relative sizes of the numbers.

Method 2 (Using algebra)
Let the required rate be *x*%.
Then, "*x*% of 18 is 44" translates to:

$$\frac{x}{100} \times \frac{18}{1} = 44$$

$$\frac{18x}{100} = \frac{44}{1}$$

$18x = 100 \times 44$ (by cross-multiplication or by multiplying both sides of the equation by 100)

$x = \frac{100 \times 44}{18}$ (by dividing both sides of the equation by 18)

$x = 244\frac{4}{9}$

$\therefore$ the required rate is $244\frac{4}{9}\%$

Example 3 A family's annual income last year was $20,000. This year, the income is $33 ,000. What is the percent increase in the annual income?

Solution

Step 1: The increase in income = $33,000 - $20,000
= $13,000

Step 2: The percent increase in income = $\frac{13000}{20000} \times 100\%$ (Finding the rate percent)
= 65%

Note: In Step 2 , the question could have been posed as: What percent of 20,000 is 13.000 ?

Example 4 On a class test , out of 20 questions, a student answered 16 questions correctly. What was the student's grade in percent?

Step 1: Fraction of questions answered correctly = $\frac{16}{20}$

Step 2: Change $\frac{16}{20}$ to percent by multiplying by 100 and attaching the % symbol.

$$\frac{16}{20} \times 100\% = 80\%$$

The student's grade was 80%.

Lesson 22 Exercises

A. (*a*) Find 80% of 25; (*b*) Find 23% of 60; (*c*) Find $5\frac{2}{9}\%$ of 1800

Answers (*a*) 20; (*b*) 13.8 or $13\frac{4}{5}$; (*c*) 94

B. 1. If 20% of a number is 72, what is the number?

2. 53 is 25% of what number?

3. 16% of what number is 82?

4. If 25% of a number is 140, what is the number?

5. If 120% of a number is 103.2, what is the number?

Answers **1.** 360 ; **2.** 212; **3.** 512.5 or $512\frac{1}{2}$; **4.** 560 ; **5.** 86.

C. 1. The monthly rent for Maria's apartment is $800. If Maria spends 25% of her monthly income on this rent, what is her monthly income?

2. A homeowner borrowed money from a bank at the interest rate of 12% per year. If the homeowner pays the bank an interest of $6, 000 per year, how much money did the homeowner borrow?

3. If $3\frac{2}{7}$ of a number is 46, What is the number?

Answer: **1.** $3,200; **2.** $50,000; **3.** 14.

D. 1. What rate percent of 32 is 12?

2. What rate% of 20 is 80?

3. On a math test, there were 40 questions. James answered 32 questions correctly. What was his grade in percent?

Answers **1.** 37.5% or $37\frac{1}{2}\%$; **2.** 400% ; **3.** 80%

Lesson 23

More Applications Involving Percent: Discount , Salary Change and Sales Tax Problems

Applications of Base Finding and Percentage Finding

In these applications, the questions are **not** worded in forms such as "find 20% of a number; "if 30% of a number is 45, what is the number?". A good approach is to reword the problem in any of these familiar forms and then proceed accordingly.

Note **1.**The fraction involved in the problem may very likely be the **rate** (but it may **not** be the rate).

Note for example that $\frac{3}{5} = 60\%$

2. The quantity following the word "of" may very likely be the **base**.
If you know any two of the three quantities, then third quantity is easily deduced.

Example 1 3 out of 5 students at a certain college study Biology.

(a) If 600 students at this college study Biology, how many students are there at this college?

(b) If one were to collect a sample of 200 students at this college, how many of these students would study Biology?

Solution:

There are a number of approaches for solving this problem.
We will cover two methods.

Note that 3 out of 5 means $\frac{3}{5}$

Part (a): We can reword this part of the question as: if $\frac{3}{5}$ of a number is 600, what is the number?

Method 1 Let the number be x

$$\text{Then } \frac{3x}{5} = 600$$
$$3x = 3000$$
$$x = 1000$$

There are 1000 students at this college.

Method 2 (see also p. 94) Divide 600 by $\frac{3}{5}$

$$600 \div \frac{3}{5} = \frac{600}{1} \times \frac{5}{3}$$
$$= 1000$$

Part (b) We can reword this part of the question as: Find $\frac{3}{5}$ of 200.

$$\frac{3}{5} \text{ of } 200 = \frac{3}{5} \times \frac{200}{1} = 120$$

That is, of a sample of 200 students, 120 students would study Biology.

Example 2

Note that since $\frac{3}{5} = 60\%$, the above problem could have been posed as 60% of students at a certain college study Biology.

(a) If 600 students study Biology, how many students are there?

(b) If one were to collect a sample of 200 students, how many would study Biology?
Solution: Proceed exactly as in Example 1 above.

Example 3 At a certain college, 53% of students registered for chemistry, and 24% registered for Biology. If there are 500 students at this college, how many students did not register for Chemistry or Biology?

Solution: We will use two methods to solve this problem.

Method 1
Step 1: Percent of students registering for Chemistry or Biology is
$(53\% + 24\%) = 77\%$

Step 2: Percent of students **not** registering for Chemistry or Biology is
$100\% - 77\% = 23\%$

Step 3: Number of students who did not register for Chemistry or Biology is

$$\begin{aligned} & 23\% \text{ of } 500 \\ &= \frac{23}{100} \times 500 \\ &= 115. \end{aligned}$$

Method 2
Step 1: Percent of students registering for Chemistry or Biology is
$(53\% + 24\%) = 77\%$

Step 2 : Find 77% of 500 and subtract the result from 500.

i.e. $\frac{77}{100} \times 500 = 385$
Then, the number of students not registering for Chemistry or Biology is

Step 3: $500 - 385 = 115$

By either Method 1 or Method 2, the number of students not registering for Chemistry or Biology is 115.

Discount Problem

Example A bag originally sold for $85.00

The selling price was reduced by 30%
What is the new selling price?

Method 1

Step 1: Find 30% of $85.00

i.e. $\frac{30}{100} \times \frac{85}{1} = 25.50$ or $\begin{array}{r} 85 \\ \times.30 \\ \hline 25.50 \end{array}$

Step 2: New Price = $85.00 - $25.50
= $59.50

Method 2 Assuming a 100% rate % for the original price,

Step 1: Subtract 30% from 100%
i.e. 100% - 30% = 70%

Step 2 : Find 70% of $85.00

i.e. $\frac{70}{100} \times \frac{85}{1} = \$59.50$ or $\begin{array}{r} 85 \\ \times.70 \\ \hline 59.50 \end{array}$

Salary Change Problem

Example

A year ago, Mary's annual salary was $45,000. This year she received a 15% raise. What is her new salary?

The approach in solving the problem is similar to that of the above discount problem, except that in this case, we add any change (in salary).

Method 1 Step 1: Find 15% of 45,000

$\frac{15}{100} \times \frac{45000}{1} = 6{,}750$

The increase (raise) in salary is $6,750

Step 2: New Salary = Original Salary + Increase in salary
= $45,000 + $6,750
= $51,750

Method 2

Step 1: Add 15% to 100%.
15% + 100 % = 115%

Step 2: Find 115% of 45,00

$\frac{115}{100} \times \frac{45000}{1} = \$51{,}750$

Sales Tax Problem

Example A book sells for $50.00
and the sales tax is 8%.
How much does the purchaser pay?

Method 1

Step 1: Find 8% of $50

$$\frac{8}{100} \times \frac{50}{1} = 4 \qquad \text{or} \qquad \begin{array}{r} 50 \\ \times.08 \\ \hline 4.00 \end{array}$$

the sales tax is $4.00

Step 2: Total cost = $50.00 + $4.00
=$54.00
∴ The purchaser pays $54.00

Method 2 Step 1: Add 8% to 100%
i.e. 8% + 100% = 108%

Step 2: Find 108% of $50.00

$$\frac{108}{100} \times \frac{50}{1} = \$54.00$$

Lesson 23 Exercises

1. A new math textbook sells for $32.00. However, a used edition of this book is sold at a 6% discount. What is the selling price of the used edition?

Answer $30.08

2, Last year , the president of a corporation was earning $160,000 per year. This year, because of financial problems, the annual salary is to be reduced to $140,000.
What is the percent decrease in salary?

Answer 12.5% or $12\frac{1}{2}$%

3. A book sells for $65.00 and the sales tax is 8.25%.
What is the total price?

Answer $70.36

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CHAPTER 1

Signed Numbers and Real Number Operations

Lesson 1: **Addition and Subtraction of Signed Numbers**
Lesson 2: **Multiplication and Division of Signed Numbers**
Lesson 3: **Operations Involving Zero, Powers and Roots of Numbers**

Lesson 1

Addition and Subtraction of Signed Numbers

Signed Numbers

A signed number is a number with either a plus sign " +" or a minus sign "-" preceding it (in front of it). If there is no sign in front of a number , we will assume that the number has a plus sign. We call a number with a plus sign a positive number, and we call a number with a minus sign a negative number. For a positive number, we sometimes do not write the plus sign, but for a negative number we **must** always write the minus sign.

The Real Number Line and Real numbers

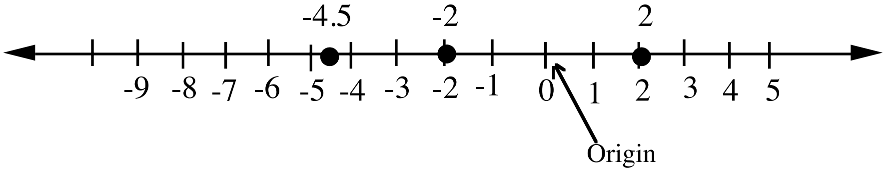

Figure 1

The real number line is a horizontal straight line with equally spaced intervals as in Figure 1 above. We label a point called the origin, 0 (zero). Points to the right of the origin are labeled positive and points to the left of the origin are labeled negative. The numbers increase as one moves from the left to the right on the real number line. Roughly speaking, a real number is a number that can be represented by a point on the real number line. The real numbers consists of the integers, fractions, mixed numbers, decimals, and radicals. In Figure 1, if the real numbers, -4.5, -2, and 2 are of interest, we can represent them by the dots shown. Every point on this line is associated with a real number; and every real number is associated with a point on this line. We can also say that the set of real numbers consists of the signed numbers and zero.

Absolute Value

The absolute value of a signed number may be defined as the number without its sign.

Examples: **1**. The absolute value of -4 , symbolized $|-4| = 4$.

2. The absolute value of 6, symbolized $|6| = 6$.

3. The absolute value of $+9$ is 9.

4. The absolute value of 0 is 0.

Note: The absolute value of a signed number is also its distance from zero on the number line.

Addition of Two Signed Numbers

To add any two signed numbers, first determine which of the following two cases is involved.

Case 1: The two numbers have the same sign

Step 1: Add the absolute values of the numbers (the unsigned parts)
Step 2: Write the common sign in front of the sum from Step 1.

Examples **1**. (+2) + (+3) = +5 or 5

2. (-3) + (-4) = -7

3. (-.5) + (-.25) = (-.75)

Case 2: The two numbers have different signs

Step 1 : Subtract the smaller absolute value from the larger absolute value
(This subtraction is exactly like the subtraction in arithmetic, where we always subtract a smaller number from a larger number)

Step 2. Write the sign of the number with the larger absolute value in front of the difference (result) from Step 1.

Examples **1**. (+5) + (-7) = -2

2. (-6) + (+9) = +3 or 3

3. (-8) + (+6) = -2

Note that **1**. -8 + 6 implies (-8) + (+6) = -2
2. -6 + 9 implies (-6) + (9) = +3 or 3
3. -8 + 9 - 7 implies that we are to add the numbers, with each number carrying the sign in front of it, and thus
-8 + 9 -7 = (-8) + (+9) + (-7) = -6

Absolute value defined more formally:

The absolute value of a real number *x* is *x* if *x* is a positive number or zero, but it is -*x* if *x* is negative number . (i.e. the negative of a negative number).

Subtraction of Signed Numbers

Procedure:

Change the sign of the number being subtracted, **and add** the changed number to the number from which you are subtracting (i.e., add the negative of the subtrahend to the minuend).

Examples

1. (+5) - (+7) (subtract) (Here, +5 is the minuend and +7 is the subtrahend)

Step 1: We change +7 to -7

Step 2: We add +5 and -7 (-7 is the changed number; it is the negative of +7)

(+5) + (-7) (add)
= - 2

2. (-6) - (-9) (subtract)

Step 1: We change -9 to +9

Step 2: We add -6 and +9 (+9 is the changed number)

(-6) + (+9) (add)
= +3

3. -7 - (-10) (subtract)

Step 1: We change -10 to + 10

Step 2: We add -7 and + 10 (+10 is the changed number)

-7 + (+10) (add)
= +3

4. 8 - (12) (subtract)

Step 1: We change 12 to -12

Step 2: We add 8 and -12 (-12 is the changed number)

8 + (-12) (add)
= -4

Note the following about the plus "+" and minus "-" signs.

The plus sign "+" may imply either the sign of a positive number, for example, +2, positive 2; or it may imply the operation of addition, for example, (+2) + (-8).

Add

The minus sign "-" may imply the sign of a negative number, for example,-6, negative 6, or the operation of subtraction, for example, (-6) - (+7).

Subtract

Note also that whenever we are to "add" we must use the rules for the addition of signed numbers and to subtract we must use the subtraction rule(s) for signed numbers.

Subtraction defined more formally:

To subtract a signed number y from another signed number x means add the negative of y to x.

that is, $x - y = x + (-y)$.

Lesson 1 Exercises

A. Add: **1.** (-3) + (-9); **2.** (+5) + (+2); **3.** (-7) + (-6)

Answers **1**. -12; **2.** +7 or 7 ; **3.** -13

B. Add: **1.** (-8) + (3); **2.** (5) + (-9); **3.** (7) + (-4)

Answers **1**. -5; **2**. -4 ; **3**. +3 or 3

C. Simplify: **1.** 5 - 7 ; **2.** - 4 - 7; **3.** -8 + 3 - 2; **4.** -3 + 8; **5.** -2 + 9 + 3;

Answers **1**. -2 ; **2**. -11 ; **3**. -7; **4**. 5 ; **5**. 10.

D. Perform the indicated operations:

1. (+7) + (+8) ; **2.** $-3\frac{2}{5}+4$; **3**. (-6) + (-8) ; **4**. 12 + (-15)

5 . (+9) + (7) ; **6**. (-3) + (4) ; **7.** 9 + (8) ; **8**. -3 + 1

9. $(-2\frac{1}{2})+(\frac{1}{4})$; **10**. - 5 + (8) ; **11**. (4.3) + (-.21); **12**. 9 - 12

13. (-6) + (-5); **14**. - 6 + 5 ; **15**. 8 - 0 ; **16.** (-3) + (-2)

17. - 4 + 8; **18**. -1 - 1; **19**. $-\frac{11}{2}+3$; **20**. 5.32 - 7.45

Also find the following:

21. $|-2|=$ **22**. $|+3|=$ **23**. $|-1\frac{1}{2}|=$ **24**. $|-5|=$

Answers: **1**. 15 ; **2**. $\frac{3}{5}$; **3**. -14 ; **4**. -3 ; **5**. 16; **6**. 1; **7**. 17 ; **8**. -2; **9**. $-2\frac{1}{4}$; **10**. 3; **11**. 4.09 ; **12**. -3 ; **13**. -11 ; **14**. -1 ; **15**. 8 ; **16**. -5 ; **17.** 4 ; **18**. -2 ; **19.** $-\frac{5}{2}$ or $-2\frac{1}{2}$; **20**. -2.13 ; **21**. 2 ; **22**. 3 ; **23** $1\frac{1}{2}$; **24.** 5

E.

Simplify the following:

1. (-5) - (-7) = **2**. (-6) - (+7) = **3**. -6 - (2) **4**. 6 - (9)

5. -1 - (-1) = **6**. $-2\frac{1}{2}-(2\frac{3}{4})$ **7**. (8) - (-10)

8. Subtract 6 from 9 **9**. Subtract 9 from 6. **10**. From 4, subtract 6

11. $\frac{3}{4}-(-\frac{1}{2}))$ **12.** (-4) - (7) **13.** (-6) - (-8) **14.** - 8 - (6)

Answers: **1**. 2 ; **2**, -13; **3**. -8 ; **4**. -3 ; **5**. 0 ; **6**. $-5\frac{1}{4}$; **7**. 18 ; **8**. 3; **9**. -3; **10**. -2; **11**. $1\frac{1}{4}$; **12**. -11 ; **13**. 2 ; **14**. -14

Lesson 2

Multiplication and Division of Signed Numbers

Multiplication of Two Signed Numbers

Implication of Multiplication Example: $2 \times 3 = 2(3) = (2)(3) = (2)\cdot(3) = (2)3 = 2\cdot 3 = 6$

Step 1: Multiply the absolute values.

Step 2: The product is positive (that is, use a plus sign or no sign) if the two numbers have the same sign; but the product is negative (that is, use a minus sign) if the two numbers have different signs.

Examples. The two numbers having the same sign

1. $(+2)(+3) = +6$ or 6

2. $(-2)(-3) = +6$ or 6

3. $-4(-2) = +8$

The two numbers having different signs

1. $(-4)(+2) = -8$

2. $(+5)(-3) = -15$

3. $-6(4) = -24$

4. $(3)(-7) = -21$

Note the following distinctions:

1. $-7 - 8 = (-7) + (-8) = -15$ (Addition)

2. $-7 - (-8) = -7 + (+8) = 1$ (Subtraction)

3. $-7(-8) = 56$ (Multiplication)

Note also that $(-3)(-2)(-4) = (+6)(-4) = -24$ (The multiplication rules are for **two** numbers at a time.)

Division of Signed Numbers

The rules of signs for division are the same as those for multiplication.

Procedure:

Step 1: Divide the absolute values of the numbers.

Step 2: The quotient is positive (that is, use a plus sign or no sign) if the two numbers have the same sign; but the quotient is negative (that is, use a minus sign) if the two numbers have different signs.

Examples: The two numbers have the same sign

1. $\frac{+8}{+4} = +2$ or 2; **2.** $\frac{-8}{-4} = +2$ or 2; **3.** $\frac{-12}{-9} = +\frac{4}{3})$ or $\frac{4}{3}$

The two numbers have different signs

1. $\frac{-6}{+2} = -3$; **2.** $\frac{8}{-2} = -4$; **3.** $\frac{-13}{3} = -\frac{13}{3}$

Reciprocals: The reciprocal of a real number A is $\frac{1}{A}$. Example : The reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$ .Thus, to find the reciprocal of a number, invert the number (or interchange the numerator and the denominator).

The reciprocal of number is also known as the multiplicative inverse of that number. The product of a number and its reciprocal is 1. Example $\frac{1}{4} \times \frac{4}{1} = 1$

Application: In $\frac{3}{7} \div \frac{9}{28}$ ($\frac{3}{7}$ is the dividend; $\frac{9}{28}$ is the divisor)

Procedure : Multiply the dividend by the **reciprocal** of the divisor. (Same as invert the divisor and multiply)

$$\frac{3}{7} \div \frac{9}{28} = \frac{3}{7} \times \frac{28}{9} = \frac{4}{3}$$

Lesson 2 Exercises

A. Simplify: **1.** (-5)(-4); **2.** -8(-9) ; **3.** 6(7) ; **4.** +5(4); **5.** (-6)(+7); **6.** -4(9)

7. -6(+3); **8.** 2(-8)

Answers **1.** 20; **2.** 72; **3.** 42; **4.** 20 ; **5.** -42; **6.** -36; **7.** -18; **8.** -16

B. Perform the indicated operations:

1. -8 - 8; **2.** -8(-8)**;** **3.** -6 - (8); **4.** -6 - (-8)**;** **5.** 9 - 7 + 3 - 8

Answers **1.** -16 ; **2.** 64; **3.** -14 **4.** +2; **5.** -3

C. Perform the indicated operations:

1. (-4)(-5) =

2. (-8)(6) =

3. 8(-7) =

4. -8(-7) =

5. -1(-5) =

6. +6(+4) =

7. 5(6) =

8. $4\frac{1}{3}\left(\frac{1}{2}\right)$) =

9. (2.3)(.5) =

10. (-4)(+2) =

11. (-4)(2) =

12. (-3)(-2)(-4) =

Review

13. - 3 - 4 =

14. 7 - (-10) =

15. -8 - 9 =

16. - 4 + (3) =

17. 3 - (-8) =

18. 4 - (9) =

19. $5\frac{3}{4} - 8\frac{1}{4}$ =

20. (-10) - (-12) =

Answers: **1.** 20 ; **2.** -48 ; **3.** -56 ; **4.** 56 ; **5.** 5 ; **6.** 24 ; **7.** 30; **8.** $2\frac{1}{6}$; **9.** 1.15 ; **10.** -8 ;
11. -8 ; **12.** -24 ; **13** -7 ; **14.** 17; **15.** -17 ; **16.** -1 ; **17.** 11 ; **18.** -5 ; **19.** -$2\frac{1}{2}$; **20.** 2

D. Simplify: **1.** (-6) ÷ (-3); **2.** $\frac{-12}{-4}$; **3.** $\frac{-4}{-6}$; **4.** (-10) ÷ (2); **5.** $\frac{24}{-6}$; **6.** $\frac{-15}{3}$

Answers **1.** 2 ; **2.** 3 ; **3.** $\frac{2}{3}$; **4.** -5 ; **5.**-4; **6.** -5

E. Carry out the indicated operations:

1. (+12) ÷ (+2); **2.** (-6) ÷ (-2); **3.** $\frac{14}{-7}$; **4.** (6) ÷ (-2); **5.** $-\frac{1}{2} \div \frac{1}{4}$;

6. Divide -9 by 2; **7.** Divide $2\frac{1}{4}$ by $-\frac{1}{4}$; **8.** Simplify: $\frac{4\frac{1}{5}}{2\frac{1}{5}}$; **9.** Simplify: $\frac{-7.5}{.3}$;

10. Simplify: $\frac{4.28}{-100}$; **11.** $\frac{+12}{-16}$ **12.** $\frac{-16}{12}$; **13.** Find the reciprocal of $\frac{4}{5}$.

Review: **14.** -4 - 2 = **15.** 8 - 3 = **16.** 3 - 8 =

17. -1 - 1 = **18.** 3 - (-4) = **19.** 4 - (12) =

Answers: **1.** 6 ; **2.** 3 ; **3.** -2 ; **4.** -3 ; **5.** -2 ; **6.** $-\frac{9}{2}$; **7.** -9; **8.** $1\frac{10}{11}$; **9.** -25 ; **10.** -.0428; **11.** $-\frac{3}{4}$;
12. $-\frac{4}{3}$ or $-1\frac{1}{3}$; **13.** $\frac{5}{4}$;**14.** -6 ; **15.** 5 ; **16.** -5 ; **17.** -2 ; **18.** 7 ; **19.** -8.

Lesson 3

Operations Involving Zero, Powers and Roots of Numbers

Operations Involving Zero

Addition

Examples 1. $6 + 0 = 6$; **2.** $0 + 4 = 4$; **3.** $-7 + 0 = -7$; **4**. $0 - 5 = -5$; **5**. $b + 0 = b$

Multiplication

Examples 1. $5(0) = 0$; **2.** $(7)(9)(0)(14) = 0$

Note: **3**. $0 \times 0 = 0$

Division

Examples 1. $\frac{0}{5} = 0$ (because $5 \times 0 = 0$)

2. $\frac{0}{-6} = 0$ (because $-6 \times 0 = 0$

But 3. $\frac{5}{0}$ is undefined.(There is **no** number such that 0 × "that number" = 5. Do not divide by 0)

4. $\frac{0}{0}$ is indeterminate. (Any number will do since 0 × "any number" = 0)

Square root: $\sqrt{0} = 0$

Powers of Signed Numbers

Examples 1. $2^5 = (2)(2)(2)(2)(2) = 32$

2^5 is read as 2 to the fifth power. (or 2 raised to the fifth power)
The exponent "5" indicates how many times the base "2" is being used as a factor.

2. $(-2)^3 = (-2)(-2)(-2)$
$= -8$

3. $(-3)^2 = (-3)(-3)$
$= 9$

Zero as an Exponent

Any (nonzero) number raised the to power zero is 1.

Examples 1. $b^0 = 1$; **2.** $4^0 = 1$; **3.** $(7x^2dz)^0 = 1$

Note that 0^0 is indeterminate. Example: $0^0 = \frac{0^2}{0^2} = \frac{0}{0}$ which is indeterminate (see above)

Roots of Signed Numbers

Root finding and Power finding are inverse operations (in much the same way as multiplication and division are inverse operations).

Square Root: Symbol " $\sqrt{\ }$ " or " $\sqrt[2]{\ }$ "

Finding the square roots of perfect squares by inspection or by factoring

1. We may cheaply define the square root (principal square root) of a nonzero number as one of the two equal **positive** factors of that number. We exclude negative roots.

2. The square root of 0 is 0.

Examples

1. The square root of 9 is symbolized $\sqrt{9}$, and

$\sqrt{9} = 3$ because $3^2 = 9$

Note that , according to the above definition, $\sqrt{9} = \sqrt{(3)(3)} = 3$

2. The square root of 64 is written $\sqrt{64}$, and

$\sqrt{64} = 8$, because $8^2 = 64$.

Also, $\sqrt{64} = \sqrt{(8)(8)} = 8$

3. Note: $\sqrt{0} = 0$, because $0^2 = 0$

If the given number is not a perfect square, we will use tables or a calculator to find the square root.

The **Cube Root** : Symbol" $\sqrt[3]{\ }$ "

From the above square root definition, we can similarly define the cube root of a number as one of the three equal factors of that number. (In this definition, we do **not** specify **positive root,** since the cube root may be positive or negative, depending on whether the given number is positive or negative.)

Examples

1. $\sqrt[3]{8} = 2$; because $2^3 = 8$. Note that $\sqrt[3]{8} = \sqrt[3]{(2)(2)(2)} = 2$

2. $\sqrt[3]{-8} = -2$; because $(-2)^3 = -8$. Note that $\sqrt[3]{(-2)(-2)(-2)} = -2$

More formal definition of square root

1. $\sqrt{A} = r$ if $r^2 = A$, where A, and r are both real and positive.

2. The square root of zero is zero (i.e. $\sqrt{0} = 0$,).

Lesson 3 Exercises

A.. Simplify: **1.** 9 + 0; **2.** 3 - 0; **3.** -6 - 0 **;** **4.** (6)(0); **5.** (0)(-7);

6. $\frac{0}{4}$; **7.** $\frac{0}{-5}$; **8.** $\frac{9}{0}$**;** **9.** 0(0) ; **10**. (78)(5)(0)(32); **11.** $\frac{0}{+6}$; **12**. 0(-8)

Answers **1.** 9; **2.** 3 ; **3**. -6 ; **4.** 0; **5.** 0 ; **6.** 0; 7. 0 ; 8. undefined; **9.** 0 ; **10.** 0 ; **11.** 0 ; **12.** 0

B. Evaluate the following:

1.3^3 **2**. 2^4 **3**. 8^2 **4**. $(-5)^3$

5. $(-1)^3$ **6**. $(-1)^4$ **7**. $(-1)^5$ **8**.$(-\frac{1}{2})^3$

9. 4^0 **10**. 0^0 **11**. 0^3 **12**. $(\frac{3}{2})^2$

13 $(-5)^0$ **14**.$(-2)^3$ **15.**$(-2)^4$

Review:

16. - 6 - 6 ; **17**. -7(-5); **18.** 9 - (-8); **19**. 9 - 8 **20** - 9 - 8

21 8 - 9; **22** $\frac{-16}{4}$ **23**. $\frac{0}{-3}$; **24**.$\frac{6}{0}$; **25**. (.6)(-.7)

Answers: **1**. 27; **2**. 16 ; **3**. 64 ; **4**. -125 ; **5**. -1 ; **6**. 1 ; **7**. -1 ; **8** $-\frac{1}{8}$; **9**. 1 ; **10**. indeterminate; **11**. 0 ; **12**. $\frac{9}{4}$; **13**. 1 ; **14**. -8 ; **15**. 16; **16**. -12; **17**. 35 ; **18**. 17 ; **19**. 1 ; **20**. -17 ; **21**. -1 ; **22**. -4 ; **23**. 0 ; **24**. undefined ; **25**. -.42

C. Evaluate the following : **1.** 4^3; **2.** $(-3)^4$; **3.** $\sqrt{64}$; **4.**$(-5)^3$; **5.**$\sqrt[3]{-64}$.

Answers **1**. 64 ; **2.** 81 ; **3.** 8 ; **4.** **-**125 ; **5.** -4

D.

Simplify the following:

1. $\sqrt{25}$ **2**. $\sqrt{81}$ **3**.$\sqrt{36}$ **4**. $\sqrt{256}$

5.$\sqrt{1024}$ **6**. $\sqrt[3]{64}$ **7**.$\sqrt[4]{81}$ **8**. $\sqrt{121}$

9 $\sqrt{\frac{25}{49}}$ **10**.$\sqrt[3]{-8}$

Review

11. - 2 - 2 **12**. -3(-2) **13**. - 4 - (-7) **14**. -2 ÷ 6

15. 3(0) **16**. 3^4 **17**.$\sqrt{100}$ **18**. 6 - (-14)

Answers: **1**. 5 ; **2**. 9 ; **3**. 6 ; **4**. 16 ; **5**. 32 ; **6**. 4 ; **7**. 3 ; **8**. 11 ; **9**.$\frac{5}{7}$; **10**. -2; **11** -4; **12**. 6 ; **13**. 3 ; **14**. $-\frac{1}{3}$; **15**. 0 ; **16**. 81 ; **17**. 10 ; **18**. 20.

Test # 1 - Student's Self-Test (**Always, Test yourself before you are tested**)

Attempt all questions on clean sheets of paper, **Do not write in the book** Show all necessary work.

1. (a) (-4) + (-7)=

(b) (-6) + (2)=

2. $-6\frac{2}{5}+4$ =

3. (a) (4.3) + (-.27)=

(b) -6 - 8 =

4. (a) (-6) - (-7) =

(b) 5 - 20 =

5. (a) Subtract 4 from 7.

(b) -5 - 12 =

6. (a) - 4 - (7)=

(b) -6(-6) =

7. (a) (-7)(-3) =

(b) (-2)(+6) =

8. (a) $\left|-4\frac{1}{2}\right|$ =

(b) - 7 - (-3)=

9. (a) 8 - 5 + 7 - 9=

(b) (-20) ÷ (5) =

10. (a) $4\frac{1}{3}\left(\frac{1}{2}\right)$=

(b) $\frac{-14}{21}$ =

11. (a) $6\frac{3}{4}-8\frac{1}{4}$=

(b) -5(-7) =

12. (a) $\frac{0}{-7}$=

(b) - 1 - 1 - 1 =

13. (a) $\frac{8}{0}$=

(b) (-5) + (-4) =

14. (a) 8^0=

(b) -3(-5) =

15. (a) $\frac{0}{-4}=$
(b) -1 - 5 =

16. (a) $\sqrt{49}=$
(b) - 4 - (2) =

17. (a) $\sqrt{\frac{16}{25}}=$
(b) 4(0)=

18. (a) $\sqrt{512}=$
(b) 7 - 3 - 6 =

19. (a) - 3 - 1 + 9 - 8 =
(b) $\frac{-8}{12}=$

20. (a) 8 - 8 =
(b) $\frac{40}{-16}=$

21. (a) (-12) ÷ (4) =
(b) (- 7) + (-8) =

22. (a) - 1 - 5 + 6 - 8 =
(b) -9 - 2 =

23. (a) 8 - (-5) =
(b) (-6) + (-9) =

24. (a) -2(-6)(-2) =
(b) -2(-6)(-2)(-3) =

25. (a) - 6 - (-8) =
(b) (+3) + (-7) =

Bonus: (a) 8 - 7 - 5 =
(b) $\frac{0}{0}=$

Answers: **1**. (a) -11; (b) -4 ; **2**. $-2\frac{2}{5}$; **3**. (a) 4.03; (b) -14 ; **4**.(a) 1; (b) -15 ; **5**. (a) 3; (b) -17; **6**. (a) -11; (b) 36 ; **7**. (a) 21; (b) -12 ; **8**. (a) $4\frac{1}{2}$; (b) -4 ; **9**. (a) 1; (b) -4 ; **10.** (a) $2\frac{1}{6}$; (b) $-\frac{2}{3}$; **11**. (a) $-1\frac{1}{2}$; (b) 35; **12**. (a) 0; (b) -3 ; **13**. (a) undefined; (b) -9 ; **14**. (a) 1; (b) 15 ;
15. (a) 0; (b) -6 ; **16**. (a) 7; (b) -6 ; **17**. (a) $\frac{4}{5}$; (b) 0; **18.** (a) $16\sqrt{2}$; (b) -2; **19** (a) -3; (b) -$\frac{2}{3}$; **20**. (a) 0; (b) $-\frac{5}{2}$; **21**. (a) -3; (b) -15; **22.** (a) -8; (b) -11; **23.** (a) 13; (b) -15 ;
24 (a) -24; (b) 72 ; **25.** (a) 2; (b) -4; **Bonus**: (a) -4; (b) indeterminate.

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CHAPTER 15

Complex Numbers

Lesson 42: **Definition, Powers of i, Square Root of Negative Numbers**
Lesson 43: **Addition and Subtraction of Complex Numbers**
Lesson 44: **Multiplication and Division of Complex Numbers**

Lesson 42

Definition, Powers of i, Square Root of Negative Numbers

Sometimes, in attempting to solve certain polynomial equations, we arrive at situations in which we have to find the square roots of negative numbers.

Example: If $x^2 = -1$, then $x = \pm\sqrt{-1}$
Since, for the set of real numbers, there is no provision for the square root of a negative number, we introduce a number "i" which we call the imaginary unit, with the following definition:

Definition: $i = \sqrt{-1}$ or $i^2 = -1$ (We will use both forms of the definition; memorize them)

For example, $(\sqrt{-1})(\sqrt{-1}) = (i)(i) = i^2 = -1$

Powers of i (Cyclical property of i or i^2) **:** All powers of i are either equal to ±1 or ± i.

Examples

(a) $i^2 = -1$

(b) $i^3 = (i^2)i$
$= -1(i)$
$= -i$

(c) $i^4 = (i^2)(i^2)$
$= (-1)(-1)$
$= 1$

(d) $i^{10} = (i^4)(i^4)(i^2)$
$= (1)(1)(-1)$
$= -1$

Note above: Even powers of i are equal to ±1. Odd powers of i are equal to ± i.

Note that the imaginary unit "i" is only a tool in mathematics, and that it is not more imaginary (in the literal sense) than the real number 3. The introduction of the imaginary unit allows us to find the square roots of negative numbers.

Example Find the square root: (a) -4 ; (b) -25 ; (c) Simplify: $\sqrt{-15}$.

Solution **(a)** $\sqrt{-4} = (\sqrt{-1})(\sqrt{4})$
$= i(2)$

Therefore $\sqrt{-4} = 2i$

(b) $\sqrt{-25} = \sqrt{-1}(\sqrt{25})$
$= i(5)$

$\sqrt{-25} = 5i$

(c) $\sqrt{-15} = (\sqrt{-1})(\sqrt{15})$
$= i\sqrt{15}$ or $\sqrt{15}\, i$

Note: In (c) we prefer the first form of the answer; because, sometimes, if we are not careful in writing "i", the "i" may look as if it is under the radical sign . However, if no radical is involved we will leave the answers as in (a) and (b) above. In some old textbooks, you may find "i" written after the radical.

Generally, $\sqrt{-b} = (\sqrt{-1})(\sqrt{b})$ $(b \geq 0)$
$= i\sqrt{b}$

The introduction of the imaginary unit helps us to expand the real number system to a more general system called the complex number system.

If we denote a complex number by z, then $z = a + b\mathrm{i}$, where a and b are real numbers; a is called the real part and b is called the imaginary part. If $b = 0$, we have a pure real number (e.g., $z = 3 + 0i = 3$). If $a = 0$, we have a pure imaginary number ($z = 0 + 2i = 2i$). Thus, the product bi is called a pure imaginary number.

Distinction Between Square Roots of Negative Numbers and Roots of Equations

Roots of numbers (Principal Roots)**:** $\sqrt{-b} = i(\sqrt{b})$ $(b \geq 0)$

Example (a) $\sqrt{-4} = (\sqrt{-1})(\sqrt{4})$
$= 2i.$

(b) $\sqrt{-25} = \sqrt{-1}(\sqrt{25})$
$= 5i.$

Roots of equations: Here, we must note that we have more than one root. (two roots.)

Example 1 Consider the solution to the equation $x^2 = -16$.

Solution $x^2 = -16$
$x = \pm\sqrt{-16}$

$= \pm 4i$, which means $x = +4i$ or $x = -4i$.

Example 2 Solve for x:: $x^2 = -4$.

Solution
$x = \pm\sqrt{-4}$
$x = \pm 2i$ (or $x = +2i$ or $x = -2i$).

Lesson 42 Exercises

Find the square root or simplify:

1. $\sqrt{-9}$; **2** $\sqrt{-49}$; **3.** $\sqrt{-36}$; **4.** $\sqrt{-18}$ **;** **5.** $\sqrt{-\frac{4}{9}}$; **6.** i^{40}; **7.** i^{93} ; **8.** i^{100}

9. $\sqrt{-28}$; **10.** $\sqrt{-32}$

Answers: **1.** $3i$; **2.** $7i$; **3.** $6i$; **4.** $3i\sqrt{2}$; **5.** $\frac{2}{3}i$; **6.** 1 ; **7.** i ; **8.** 1 ; **9.** $2i\sqrt{7}$; **10.** $4i\sqrt{2}$

Lesson 43

Addition and Subtraction of Complex Numbers

In adding complex numbers, we will add the real parts, and then add the imaginary parts (in much the same way as we add like terms in polynomial addition).

Perform the indicated operations, leaving the answers in the form $a + bi$.

Example 1 Simplify: $(-3 + 5i) + (-2 +7i)$

Step 1: Remove the parentheses.

$$(-3 + 5i) + (-2 + 7i)$$

$$= -3 + 5i - 2 + 7i$$

Step 2: Add the real parts, and add the imaginary parts.

$$-3 + 5i - 2 + 7i$$
$$= - 5 + 12i \ .$$

Scrapwork:
For the real parts: $-3 - 2 = - 5$
For the imaginary parts: $5+ 7 = 12$

Example 2 Simplify: $(6 - 5i) - (-3 + 2i)$

Solution Remove the parentheses and add.

$$(6 - 5i) - (-3 + 2i)$$
$$= 6 - 5i + 3 - 2i$$

$$= 9 - 7i$$

Example 3 Simplify: $(5 + 2i) + (-3 - 6i)$

Solution

Remove the parentheses and add.
$(5 + 2i) + (-3 - 6i)$
$= 5 + 2i - 3 - 6i$
$= 5 - 3 + 2i - 6i$ <-------you may skip this step.
$= 2 - 4i$

or adding vertically:

$$\begin{array}{r} 5 + 2i \\ \underline{-3 - 6i} \\ 2 - 4i \end{array} \qquad \text{(Adding).}$$

Example 4 Simplify : $(3 + 4i\} - (2 - 5i)$ (subtraction)

Solution

Remove the parentheses and add.

$(3 + 4i) - (2 - 5i)$
$= 3 + 4i - 2 + 5i$
$= 3 - 2 + 4i + 5i$ <-------you may skip this step.
$= 1 + 9i$

Example 5 Simplify : $(6 - 3i) + (2 + 3i)$

Solution

$(6 - 3i) + (2 + 3i)$
$= 6 - 3i + 2 + 3i$
$= 8 + 0i$
$= 8$

Lesson 43 Exercises

Simplify: **1.** $4 + 3i - 6 + 5i$; **2.** $(7 - 8i) + (2 + 9i)$ **3.** $(2 - 3i) -(7 - 4i)$;

4. Subtract $(1 - i)$ from $4 - 2i$; **5.** $4 + 2i - 3(4 - 6i) + i$; **6.** $- i^5 + i^2$; **7.** $4 - i + i^2$

Answers: **1.** $-2 + 8i$; **2.** $9 + i$; **3.** $-5 + i$; **4.** $3 - i$; **5.** $-8 + 21i$; **6.** $-1 - i$; **7.** $3 - i$

Lesson 44

Multiplication and Division of Complex Numbers

Multiplication of Complex Numbers

The approach here is multiply, replace i^2 by -1, (or higher powers of i by ± 1 or $\pm i$) add the real parts and add the imaginary parts.

Example 1 Multiply $-4 - 2i$ and $-5 + i$

Solution

Step 1: Multiply as you multiply binomials

$(-4 - 2i)(-5 + i)$

$= -4(-5 + i) + (-2i)(-5 + i)$ <--- You may skip this step.

$= 20 - 4i + 10i - 2i^2$

Step 2: (Replace i^2 by -1, since $i^2 = -1$ by definition)

$= 20 - 4i + 10i - 2(-1)$

$= 20 + 6i + 2$

Step 3: (Add the like terms: add the real parts; and add the imaginary parts)

$= 22 + 6i$

Example 2 Multiply $3 + 2i$ and $4 + 5i$

Solution

Procedure: Multiply, replace i^2 by -1, and simplify.

$(3 + 2i)(4 + 5i)$

$= 12 + 15i + 8i + 10i^2$

$= 12 + 23i + 10(-1)$

$= 12 + 23i - 10$

$= 2 + 23i$ (Adding)

Example 3 Simplify : $4(6 - 3i)$

Solution

$4(6 - 3i)$

$= 24 - 12i$

Example 4 Simplify: $(4 + 2i)(3 - 6i)$

Solution The above implies multiplication.

Multiply, replace i^2 by -1, and add the like terms.

$(4 + 2i)(3 - 6i)$

$= 12 - 24i + 6i - 12i^2$

$= 12 - 24i + 6i - 12(-1)$

$= 12 - 24i + 6i + 12$

$= 24 - 18i$ (Note: $12 + 12 = 24$ and $-24i + 6i = -18i$)

Example 5 Simplify: $(7 + 2i)(7 - 2i)$

Solution

$(7 + 2i)(7 - 2i)$
$= 49 - 14i + 14i - 4i^2$
$= 49 - 4(-1)$
$= 49 + 4$
$= 53$

Multiplication of the square roots of negative numbers

Example 1 Find the product of $(\sqrt{-8})$ and $(\sqrt{-2})$

Solution

Step 1: Change to complex number forms.

$\sqrt{-8} = \sqrt{-1}(\sqrt{8})$
$= i\sqrt{8}$

Similarly , $\sqrt{-2} = i\sqrt{2}$

Step 2: Multiply the complex forms now.

$(\sqrt{-8})(\sqrt{-2}) = i\sqrt{8}\, i\sqrt{2}$
$= i^2\sqrt{16}$ $(\sqrt{16} = 4)$
$= i^2 (4)$
$= (-1)(4)$ $(i^2 = -1)$
$= - 4$

In the above problem (Example 1), you may skip Step 1 and show only Step 2.

Example 2 Find the product $(\sqrt{-49})(\sqrt{-25})$

Solution

Step 1: Change to complex number forms.

Then, $\sqrt{-49} = i\sqrt{49} = i(7) = 7i$

<---You may do Step 1 mentally and show only Step 2.

$\sqrt{-25} = i\sqrt{25} = i(5) = 5i$

Step 2: Multiply the complex forms now.

Then, $(\sqrt{-49})(\sqrt{-25}) = (7i)(5i)$
$= (7)(5)i^2$
$= 35(-1)$ $(i^2 = -1)$
$= -35$

We must note in the last two examples that it was necessary first to express the square roots of the negative numbers in terms of *i* before proceeding to multiply. Failure to do this may result in error such as the following:

Wrong procedure---> $(\sqrt{-8}(\sqrt{-2}) = \sqrt{(-2)(-8)}$
$= \sqrt{16}$
$= 4$, **which is a wrong answer**.

Generally, it is true that $(\sqrt{a})(\sqrt{b}) = \sqrt{ab}$ if *a* and *b* are positive but it is not true if *a* and *b* are negative.

Complex conjugates

Definition : The **conjugate** of a given binomial is another binomial that differs from the given binomial only in the sign of one of the terms. The conjugate of $a + b$ is $a - b$; and the conjugate of $a - b$ is $a + b$.

The **conjugate** of $a + bi$ is $a - bi$. The conjugate of $a - bi$ is $a + bi$. A complex number and its conjugate differ only in the sign of the imaginary part. To find the conjugate of a complex number, change the sign of the imaginary part and keep the sign of the real part unchanged.

Examples:

The conjugate of $2 + 3i$ is $2 - 3i$. The conjugate of $4 - 2i$ is $4 + 2i$. The conjugate of $7 - 5i$ is $7 + 5$. The conjugate of $-3 + 6i$ is $-3 - 6i$. The conjugate of $-2 -3i$ is $-2 + 3i$. The conjugate of $4i$ is $- 4i$. The conjugate of 7 is 7

The product of a complex number and its conjugate is a real number.

The product of $-2 -3i$ and $-2 + 3i$. $=4 - 6i + 6i - 9i^2 = 4 - 9(-1) = 4 + 9 =$ **13**, a real number.

The product of $a + bi$ and $a - bi = a^2 + b^2$

We can use the conjugate of a complex number to rationalize the denominator of a fraction or to divide by a complex number.

Division of Complex Numbers

Example 1 Simplify : $\dfrac{2 + 3i}{4 - 5i}$ or divide $2 + 3i$ by $4 - 5i$

To simplify the above complex number is meant we are to write it in the form $z = a + bi$.

The operation here is similar to that of the rationalization of denominators of radical expressions.

To simplify the above expression, we **multiply both the denominator and the numerator by the conjugate of the denominator.**

Solution The conjugate of $4 - 5i$ is **$4 + 5i$.** (See also the note below)

Multiply both the denominator and the numerator by $4 + 5i$.

Then, we obtain:

$$\frac{2 + 3i}{4 - 5i}$$
$$= \frac{(2 + 3i)}{(4 - 5i)} \cdot \frac{\mathbf{(4 + 5i)}}{\mathbf{(4 + 5i)}}$$
$$= \frac{8 + 10i + 12i + 15i^2}{16 + 20i - 20\mathrm{i} - 25i^2}$$
$$= \frac{8 + 22i + 15(-1)}{16 + 0 - 25(-1)}$$
$$= \frac{8 + 22i - 15}{16 + 25}$$
$$= \frac{8 + 22i - 15}{41}$$
$$= \frac{-7 + 22i}{41}$$
$$= -\frac{7}{41} + \frac{22}{41}i$$

We may observe above that the **denominator** does **not** contain the imaginary unit "i", even though the numerator contains the imaginary unit.

Note above that we could have multiplied by $-4 - 5i$. (Try it.) It is therefore not critical in this problem which terms should differ in sign, (so far as the rationalization is concerned) provided that either the real parts differ in sign or the imaginary parts differ in sign, but **not** both.

We may produce a minus sign in the denominator which we can take care of as usual.

Example 2 Simplify: $\dfrac{5 + 4i}{-3 + 2i}$

Solution Multiply both the denominator and the numerator by -3 - 2i (The conjugate of -3 + 2i).

$$\frac{5 + 4i}{-3 + 2i}$$

$$= \frac{(5 + 4i)}{(-3 + 2i)} \cdot \frac{\mathbf{(-3 - 2}\boldsymbol{i}\mathbf{)}}{\mathbf{(-3 - 2}\boldsymbol{i}\mathbf{)}}$$

$$= \frac{-15 - 10i - 12i - 8i^2}{9 + 6i - 6i - 4i^2}$$

$$= \frac{-15 - 10i - 12i - 8(-1)}{9 + 0 - 4(-1)}$$

$$= \frac{-15 - 22i + 8}{9 + 4}$$

$$= \frac{-7 - 22i}{13}$$

$$= -\frac{7}{13} - \frac{22}{13}i \qquad (-15 + 8 = -7)$$

Example 3 Simplify: $\dfrac{4 - 2i}{i}$

Solution

$$\frac{4 - 2i}{i}$$

$$= \frac{(4 - 2i)}{i} \frac{\mathbf{(-}\boldsymbol{i}\mathbf{)}}{\mathbf{(-}\boldsymbol{i}\mathbf{)}} \qquad \text{(The conjugate of } i \text{ is } -i \text{ ; but you could also use } i\text{)}$$

$$= \frac{(4 - 2i)(-\mathrm{i})}{(i)(-i)}$$

$$= \frac{-4i + 2i^2}{-i^2}$$

$$= \frac{-4i + 2(-1)}{-(-1)} \qquad (i^2 = -1)$$

$$= \frac{-4i - 2}{+1}$$

$$= -2 - 4i$$

Lesson 44 Exercises

A **1**. Multiply: $4 + 3i$ and $2 + 5i$; **2.** Multiply $2 - 5i$ and $-2 + 6i$; **3.** Simplify: $(6 -7i)(2 + 3i)$; **4.** Simplify $(1 - i)(1 + i)$; **5**. $(2 + 3i)^2$; **6**. $(4i^2)(-8i)(i^2)$; **7**. $(5 + 3i)(5 - 3i)$

Answers: **1.** $-7 + 26i$; **2.** $26 + 22i$; **3.** $33 + 4i$; **4.** 2 ; **5**. $-5 + 12i$; **6**. $-32i$; **7**. 34.

B Divide: **1**. $\dfrac{8 - 6i}{2}$; **2**. $\dfrac{6 + 8i}{2i}$; **3**. $\dfrac{\sqrt{16}}{\sqrt{-16}}$; **4**. $\dfrac{3}{4 - \sqrt{-9}}$; **5**. $\dfrac{4 - 2i}{3 + 5i}$

Answers: **1**. $4 - 3i$; **2**. $4 - 3i$; **3**. $-i$; **4**. $\dfrac{12}{25} + \dfrac{9}{25}i$; **5**. $\dfrac{1}{17} - \dfrac{13}{17}i$

C Find the product of each of the following:

1. $\left(\sqrt{-4}\right)\left(\sqrt{-1}\right)$; **2**. $\left(\sqrt{-16}\right)\left(\sqrt{-4}\right)$; **3**. $\left(\sqrt{-25}\right)\left(\sqrt{49}\right)$; **4**. $\left(\sqrt{-8}\right)\left(\sqrt{-2}\right)$

Answers: **1**. -2; **2**. -8; **3**. $35i$; **4**. -4.

D Simplify: **1**. $\dfrac{2 + 3i}{5 - 2i}$; **2.** $\dfrac{4}{3 - 4i}$; **3.** $\dfrac{5 + 2i}{-i}$ **4.** Divide $4 + 3i$ by $-2 + 3i$;

Simplify: **5**. $\dfrac{2 - 6i}{4 + 3i}$; **6**. $\dfrac{i - 2}{2 - i}$; **7**. $(3 - 2i)(4 + 2i)(i^2)$; **8**. $(6 - 4i)(6 + 4i)$

Answers: 1. $\frac{4}{29} + \frac{19}{29}i$; **2**. $\frac{12}{25} + \frac{16}{25}i$; **3**. $-2 + 5i$; **4**. $\frac{1}{13} - \frac{18}{13}i$; **5**. $-\frac{2}{5} - \frac{6}{5}i$; **6**. -1 ; **7**. $-16 + 2i$; **8**. 52

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CHAPTER 15

Complex Numbers

Lesson 42

Definition, Powers of i, Square Root of Negative Numbers

Sometimes, in attempting to solve certain polynomial equations, we arrive at situations in which we have to find the square roots of negative numbers.

Example: If $x^2 = -1$, then $x = \pm\sqrt{-1}$
Since, for the set of real numbers, there is no provision for the square root of a negative number, we introduce a number "i" which we call the imaginary unit, with the following definition:

Definition: $i = \sqrt{-1}$ or $i^2 = -1$ (We will use both forms of the definition; memorize them)

For example, $(\sqrt{-1})(\sqrt{-1}) = (i)(i) = i^2 = -1$

Powers of i (Cyclical property of i or i^2) **:** All powers of i are either equal to ±1 or ± i.
Examples

(a) $i^2 = -1$
(b) $i^3 = (i^2)i$
$= -1(i)$
$= -i$

(c) $i^4 = (i^2)(i^2)$
$= (-1)(-1)$
$= 1$
(d) $i^{10} = (i^4)(i^4)(i^2)$
$= (1)(1)(-1)$
$= -1$

Note above: Even powers of i are equal to ±1. Odd powers of i are equal to ± i.

Note that the imaginary unit "i" is only a tool in mathematics, and that it is not more imaginary (in the literal sense) than the real number 3. The introduction of the imaginary unit allows us to find the square roots of negative numbers.

Example Find the square root: (a) -4 ; (b) -25 ; (c) Simplify: $\sqrt{-15}$.

Solution **(a)** $\sqrt{-4} = (\sqrt{-1})(\sqrt{4})$
$= i(2)$

Therefore $\sqrt{-4} = 2i$

(b) $\sqrt{-25} = \sqrt{-1}(\sqrt{25})$
$= i(5)$

$\sqrt{-25} = 5i$

(c) $\sqrt{-15} = (\sqrt{-1})(\sqrt{15})$
$= i\sqrt{15}$ or $\sqrt{15}\, i$

Note: In (c) we prefer the first form of the answer; because, sometimes, if we are not careful in writing "i", the "i" may look as if it is under the radical sign . However, if no radical is involved we will leave the answers as in (a) and (b) above. In some old textbooks, you may find "i" written after the radical.

Generally, $\sqrt{-b} = (\sqrt{-1})(\sqrt{b})$ $(b \geq 0)$
$= i\sqrt{b}$

The introduction of the imaginary unit helps us to expand the real number system to a more general system called the complex number system.

If we denote a complex number by z, then $z = a + b\mathrm{i}$, where a and b are real numbers; a is called the real part and b is called the imaginary part. If $b = 0$, we have a pure real number (e.g., $z = 3 + 0i = 3$). If $a = 0$, we have a pure imaginary number ($z = 0 + 2i = 2i$). Thus, the product bi is called a pure imaginary number.

Distinction Between Square Roots of Negative Numbers and Roots of Equations

Roots of numbers (Principal Roots)**:** $\sqrt{-b} = i(\sqrt{b})$ $(b \geq 0)$

Example (a) $\sqrt{-4} = (\sqrt{-1})(\sqrt{4})$
$= 2i$.

(b) $\sqrt{-25} = \sqrt{-1}(\sqrt{25})$
$= 5i$.

Roots of equations: Here, we must note that we have more than one root. (two roots.)

Example 1 Consider the solution to the equation $x^2 = -16$.

Solution $x^2 = -16$
$x = \pm\sqrt{-16}$

$= \pm 4i$, which means $x = +4i$ or $x = -4i$.

Example 2 Solve for x:: $x^2 = -4$.

Solution
$x = \pm\sqrt{-4}$
$x = \pm 2i$ (or $x = +2i$ or $x = -2i$).

Lesson 42 Exercises

Find the square root or simplify:

1. $\sqrt{-9}$; **2** $\sqrt{-49}$; **3**. $\sqrt{-36}$; **4.** $\sqrt{-18}$ **;** **5.** $\sqrt{-\frac{4}{9}}$; **6.** i^{40}; **7**. i^{93} ; **8.** i^{100}

9. $\sqrt{-28}$; **10**. $\sqrt{-32}$

Answers: **1**. $3i$; **2**. $7i$; **3.** $6i$; **4.** $3i\sqrt{2}$; **5**. $\frac{2}{3}i$; **6**. 1 ; **7**. i ; **8**. 1 ; **9**. $2i\sqrt{7}$; **10**. $4i\sqrt{2}$

Lesson 43

Addition and Subtraction of Complex Numbers

In adding complex numbers, we will add the real parts, and then add the imaginary parts (in much the same way as we add like terms in polynomial addition).

Perform the indicated operations, leaving the answers in the form $a + bi$.

Example 1 Simplify: $(-3 + 5i) + (-2 +7i)$

Step 1: Remove the parentheses.

$$(-3 + 5i) + (-2 + 7i)$$

$$= -3 + 5i - 2 + 7i$$

Step 2: Add the real parts, and add the imaginary parts.

$$-3 + 5i - 2 + 7i$$
$$= - 5 + 12i \text{ .}$$

Scrapwork:
For the real parts: $-3 - 2 = - 5$
For the imaginary parts: $5+ 7 = 12$

Example 2 Simplify: $(6 - 5i) - (-3 + 2i)$

Solution Remove the parentheses and add.

$$(6 - 5i) - (-3 + 2i)$$
$$= 6 - 5i + 3 - 2i$$

$$= 9 - 7i$$

Example 3 Simplify: $(5 + 2i) + (-3 - 6i)$

Solution

Remove the parentheses and add.

$(5 + 2i) + (-3 - 6i)$
$= 5 + 2i - 3 - 6i$
$= 5 - 3 + 2i - 6i$ <-------you may skip this step.
$= 2 - 4i$

or adding vertically:

$$\begin{array}{r} 5 + 2i \\ \underline{-3 - 6i} \\ 2 - 4i \end{array} \qquad \text{(Adding).}$$

Example 4 Simplify : $(3 + 4i\} - (2 - 5i)$ (subtraction)

Solution

Remove the parentheses and add.

$(3 + 4i) - (2 - 5i)$
$= 3 + 4i - 2 + 5i$
$= 3 - 2 + 4i + 5i$ <-------you may skip this step.
$= 1 + 9i$

Example 5 Simplify : $(6 - 3i) + (2 + 3i)$

Solution

$(6 - 3i) + (2 + 3i)$
$= 6 - 3i + 2 + 3i$
$= 8 + 0i$
$= 8$

Lesson 43 Exercises

Simplify: **1.** $4 + 3i - 6 + 5i$; **2**. $(7 - 8i) + (2 + 9i)$ **3.** $(2 - 3i) -(7 - 4i)$;

4. Subtract $(1 - i)$ from $4 - 2i$; **5.** $4 + 2i - 3(4 - 6i) + i$; **6**. $- i^5 + i^2$; **7**. $4 - i + i^2$

Answers: **1**. $-2 + 8i$; **2**. $9 + i$; **3**. $-5 + i$; **4**. $3 - i$; **5**. $-8 + 21i$; **6**. $-1 - i$; **7**. $3 - i$

Lesson 44

Multiplication and Division of Complex Numbers

Multiplication of Complex Numbers

The approach here is multiply, replace i^2 by -1, (or higher powers of i by ±1 or ±i) add the real parts and add the imaginary parts.

Example 1 Multiply -4 - 2i and -5 + i

Solution

Step 1: Multiply as you multiply binomials

$(-4 - 2i)(-5 + i)$

$= -4(-5 + i) + (-2i)(-5 + i)$ <--- You may skip this step.

$= 20 - 4i + 10i - 2i^2$

Step 2: (Replace i^2 by -1, since $i^2 = -1$ by definition)

$= 20 - 4i +10i - 2(-1)$

$= 20 + 6i + 2$

Step 3: (Add the like terms: add the real parts; and add the imaginary parts)

$= 22 + 6i$

Example 2 Multiply 3 + 2i and 4 + 5i

Solution

Procedure: Multiply, replace i^2 by -1, and simplify.

$(3 + 2i)(4 + 5i)$

$= 12 + 15i + 8i + 10i^2$

$= 12 + 23i + 10(-1)$

$= 12 + 23i - 10$

$= 2 + 23i$ (Adding)

Example 3 Simplify : 4(6 - 3i)

Solution

$4(6 - 3i)$

$= 24 - 12i$

Example 4 Simplify: (4 + 2i)(3 - 6i)

Solution The above implies multiplication.

Multiply, replace i^2 by -1, and add the like terms.

$(4 + 2i)(3 - 6i)$

$= 12 - 24i + 6i - 12i^2$

$= 12 - 24i + 6i - 12(-1)$

$= 12 - 24i + 6i + 12$

$= 24 - 18i$ (Note: 12 +12 =24 and -24i + 6i = -18i)

Example 5 Simplify: $(7 + 2i)(7 - 2i)$

Solution

$(7 + 2i)(7 - 2i)$
$= 49 - 14i + 14i - 4i^2$
$= 49 - 4(-1)$
$= 49 + 4$
$= 53$

Multiplication of the square roots of negative numbers

Example 1 Find the product of $(\sqrt{-8})$ and $(\sqrt{-2})$

Solution

Step 1: Change to complex number forms.

$\sqrt{-8} = \sqrt{-1}(\sqrt{8})$
$= i\sqrt{8}$

Similarly , $\sqrt{-2} = i\sqrt{2}$

Step 2: Multiply the complex forms now.

$(\sqrt{-8})(\sqrt{-2}) = i\sqrt{8}\ i\sqrt{2}$
$= i^2\sqrt{16}$ $(\sqrt{16} = 4)$
$= i^2\ (4)$
$= (-1)(4)$ $(i^2 = -1)$
$= -4$

In the above problem (Example 1), you may skip Step 1 and show only Step 2.

Example 2 Find the product $(\sqrt{-49})(\sqrt{-25})$

Solution

Step 1: Change to complex number forms.

Then, $\sqrt{-49} = i\sqrt{49} = i(7) = 7i$

<---You may do Step 1 mentally and show only Step 2.

$\sqrt{-25} = i\sqrt{25} = i(5) = 5i$

Step 2: Multiply the complex forms now.

Then, $(\sqrt{-49})(\sqrt{-25}) = (7i)(5i)$
$= (7)(5)i^2$
$= 35(-1)$ $(i^2 = -1)$
$= -35$

We must note in the last two examples that it was necessary first to express the square roots of the negative numbers in terms of *i* before proceeding to multiply. Failure to do this may result in error such as the following:

Wrong procedure---> $(\sqrt{-8}(\sqrt{-2}) = \sqrt{(-2)(-8)}$
$= \sqrt{16}$
$= 4$, **which is a wrong answer**.

Generally, it is true that $(\sqrt{a})(\sqrt{b}) = \sqrt{ab}$ if *a* and *b* are positive but it is not true if *a* and *b* are negative.

Complex conjugates

Definition : The **conjugate** of a given binomial is another binomial that differs from the given binomial only in the sign of one of the terms. The conjugate of $a + b$ is $a - b$; and the conjugate of $a - b$ is $a + b$.

The **conjugate** of $a + bi$ is $a - bi$. The conjugate of $a - bi$ is $a + bi$. A complex number and its conjugate differ only in the sign of the imaginary part. To find the conjugate of a complex number, change the sign of the imaginary part and keep the sign of the real part unchanged.

Examples:

The conjugate of $2 + 3i$ is $2 - 3i$. The conjugate of $4 - 2i$ is $4 + 2i$. The conjugate of $7 - 5i$ is $7 + 5$. The conjugate of $-3 + 6i$ is $-3 - 6i$. The conjugate of $-2 -3i$ is $-2 + 3i$. The conjugate of $4i$ is $- 4i$. The conjugate of 7 is 7

The product of a complex number and its conjugate is a real number.

The product of $-2 -3i$ and $-2 + 3i$. $=4 - 6i + 6i - 9i^2 = 4 - 9(-1) = 4 + 9 =$ **13**, a real number.

The product of $a + bi$ and $a - bi = a^2 + b^2$

We can use the conjugate of a complex number to rationalize the denominator of a fraction or to divide by a complex number.

Division of Complex Numbers

Example 1 Simplify : $\dfrac{2 + 3i}{4 - 5i}$ or divide $2 + 3i$ by $4 - 5i$

To simplify the above complex number is meant we are to write it in the form $z = a + bi$.

The operation here is similar to that of the rationalization of denominators of radical expressions.

To simplify the above expression, we **multiply both the denominator and the numerator by the conjugate of the denominator.**

Solution The conjugate of $4 - 5i$ is $\mathbf{4 + 5}\boldsymbol{i}$**.** (See also the note below)

Multiply both the denominator and the numerator by $4 + 5i$.

Then, we obtain:

$$\frac{2 + 3i}{4 - 5i}$$

$$= \frac{(2 + 3i)}{(4 - 5i)} \cdot \frac{\mathbf{(4 + 5}\boldsymbol{i}\mathbf{)}}{\mathbf{(4 + 5}\boldsymbol{i}\mathbf{)}}$$

$$= \frac{8 + 10i + 12i + 15i^2}{16 + 20i - 20\mathrm{i} - 25i^2}$$

$$= \frac{8 + 22i + 15(-1)}{16 + 0 - 25(-1)}$$

$$= \frac{8 + 22i - 15}{16 + 25}$$

$$= \frac{8 + 22i - 15}{41}$$

$$= \frac{-7 + 22i}{41}$$

$$= -\frac{7}{41} + \frac{22}{41}i$$

We may observe above that the **denominator** does **not** contain the imaginary unit "i", even though the numerator contains the imaginary unit.

Note above that we could have multiplied by $-4 - 5i$. (Try it.) It is therefore not critical in this problem which terms should differ in sign, (so far as the rationalization is concerned) provided that either the real parts differ in sign or the imaginary parts differ in sign, but **not** both.

We may produce a minus sign in the denominator which we can take care of as usual.

Example 2 Simplify: $\frac{5 + 4i}{-3 + 2i}$

Solution Multiply both the denominator and the numerator by -3 - 2i (The conjugate of -3 + 2i).

$$\frac{5 + 4i}{-3 + 2i}$$
$$= \frac{(5 + 4i)}{(-3 + 2i)} \cdot \frac{\mathbf{(-3 - 2}\boldsymbol{i}\mathbf{)}}{\mathbf{(-3 - 2}\boldsymbol{i}\mathbf{)}}$$
$$= \frac{-15 - 10i - 12i - 8i^2}{9 + 6i - 6i - 4i^2}$$
$$= \frac{-15 - 10i - 12i - 8(-1)}{9 + 0 - 4(-1)}$$
$$= \frac{-15 - 22i + 8}{9 + 4}$$
$$= \frac{-7 - 22i}{13}$$
$$= -\frac{7}{13} - \frac{22}{13}i \qquad (-15 + 8 = -7)$$

Example 3 Simplify: $\frac{4 - 2i}{i}$

Solution $\frac{4 - 2i}{i}$

$$= \frac{(4 - 2i)}{i}\frac{\boldsymbol{(-i)}}{\boldsymbol{(-i)}} \quad \text{(The conjugate of } i \text{ is } -i \text{ ; but you could also use } i\text{)}$$
$$= \frac{(4 - 2i)(-\mathrm{i})}{(i)(-i)}$$
$$= \frac{-4i + 2i^2}{-i^2}$$
$$= \frac{-4i + 2(-1)}{-(-1)} \qquad (i^2 = -1)$$
$$= \frac{-4i - 2}{+1}$$
$$= -2 - 4i$$

Lesson 44 Exercises

A **1**. Multiply: $4 + 3i$ and $2 + 5i$; **2.** Multiply $2 - 5i$ and $-2 + 6i$; **3.** Simplify: $(6 -7i)(2 + 3i)$;
4. Simplify $(1 - i)(1 + i)$; **5**. $(2 + 3i)^2$; **6**. $(4i^2)(-8i)(i^2)$; **7**. $(5 + 3i)(5 - 3i)$

Answers: **1.** $-7 + 26i$; **2.** $26 + 22i$; **3.** $33 + 4i$; **4.** 2 ; **5**. $-5 + 12i$; **6**. $-32i$; **7**. 34.

B Divide: **1**. $\frac{8 - 6i}{2}$; **2**. $\frac{6 + 8i}{2i}$; **3**. $\frac{\sqrt{16}}{\sqrt{-16}}$; **4**. $\frac{3}{4 - \sqrt{-9}}$; **5**. $\frac{4 - 2i}{3 + 5i}$

Answers: **1**. $4 - 3i$; **2**. $4 - 3i$; **3**. $-i$; **4**. $\frac{12}{25} + \frac{9}{25}i$; **5**. $\frac{1}{17} - \frac{13}{17}i$

C Find the product of each of the following:

1. $\left(\sqrt{-4}\right)\left(\sqrt{-1}\right)$; **2**. $\left(\sqrt{-16}\right)\left(\sqrt{-4}\right)$; **3**. $\left(\sqrt{-25}\right)\left(\sqrt{49}\right)$; **4**. $\left(\sqrt{-8}\right)\left(\sqrt{-2}\right)$

Answers: **1**. -2; **2**. -8; **3**. $35i$; **4**. -4.

D Simplify: **1**. $\frac{2+3i}{5-2i}$; **2.** $\frac{4}{3-4i}$; **3.** $\frac{5+2i}{-i}$ **4.** Divide $4+3i$ by $-2+3i$;

Simplify: **5**. $\frac{2-6i}{4+3i}$; **6**. $\frac{i-2}{2-i}$; **7**. $(3-2i)(4+2i)(i^2)$; **8**. $(6-4i)(6+4i)$

Answers: 1. $\frac{4}{29}+\frac{19}{29}i$; **2**. $\frac{12}{25}+\frac{16}{25}i$; **3**. $-2+5i$; **4**. $\frac{1}{13}-\frac{18}{13}i$; **5**. $-\frac{2}{5}-\frac{6}{5}i$; **6**. -1 ; **7**. $-16+2i$; **8**. 52

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CHAPTER 6

FUNCTIONS

Lesson 9

Sets, Relations, Functions, Comparison of Relations and Functions

Ordered Pair

An **ordered pair** of numbers is an arrangement of two numbers in a specified order. In an *x-y* rectangular coordinate system of axes, the first element (or component) is the *x*-value and the second element is the *y*-value.

Example 1 (*a*) (1, 2) <--- ($x = 1, y = 2$)
(*b*) (2, 3) <---- ($x = 2, y = 3$)
(*c*) (5, -1) <---- ($x = 5, y = -1$)

Note that each ordered pair represents a point in an *x-y* coordinate system of axes.

Set of numbers

A **set of numbers** is a well-defined collection of numbers. The numbers are called the elements or members of the set.

Example 2: If we denote the set of the numbers 2, 5 and 6 by A, then we may write $A = \{2, 5, 6\}$

Example 3: The set B of the ordered pairs (1, 2), (2, 3), and (5,-1) is given by
$B = \{(1, 2), (2, 3), (5,-1)\}$.

Relation

A **relation** is a set of ordered pairs. (A collection of ordered pairs of numbers)

Example 4: The set $E = \{(2, 3), (2, 5), (4, 6)\}$ is a relation, <---There are three ordered pairs.

Example 5: The set $C = \{(6, 2), (7,4), (11, 5)\}$ is a relation.

Definition of a **Function**

A function may be defined in a number of ways, namely,

(a) in terms of ordered pairs; (b) in terms of a rule involving two variables;
(c) in terms of a rule for inputs and outputs; (d) in terms of correspondence of two sets; (e) as a graph

Definition 1: In terms of ordered pairs

A **function** is a relation in which no two distinct ordered pairs have the same first component; or a function is a set of ordered pairs in which for any two different ordered pairs, the first elements are different. The set in Example 5, above, is a function but the set in Example 4 is not a function, because the first two ordered pairs have the same first element, namely 2.

The set of all the first elements of the ordered pairs is called the **domain** of the function; and the set of all the second elements is called the **range** of the function.

Example 6: In the function $C = \{(6, 2), (7, 4), (11, 5)\}$. The domain, D = {6, 7, 11} (first elements)
. The range, R = {2, 4, 5}. (second elements)

Other definitions of a function

Definition 2: If x and y are two variables. then we say that y is a function of x if there is a rule which gives just one corresponding value of y for **each** value of x. The variable x is called the independent variable, and a variable y is called the dependent variable. The rule may be specified in the form of a set, in the form of a graph, in the form of a table, or in the form of an equation or formula.

The **domain** of a function is the set of numbers that can be assigned to x (the independent variable).

The **range** of a function is the set of all the corresponding numbers y (the dependent variable) associated by the function (rule) with the numbers, x, in the domain .

We symbolize that f is a function of x by $f(x)$, where x is called the independent variable, y is called the dependent variable. **Note**: $f(x)$s is read as f of x.

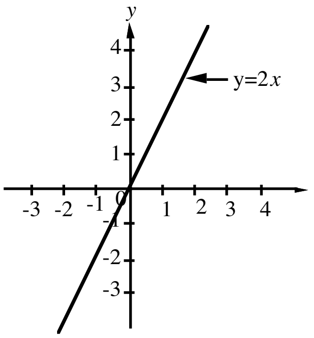
The following are examples of how the rules for functions may be specified:

(a) In the form of an equation or a formula: $y = 2x$.
(b) In the form of a set: $\{(2,3),(1,4),(7,5)\}$.
(c) In the form of a table for x and y: See Table 1.
(d) In the form of a graph. See Figure

Table 1:: $y = 2x$

$x =$	0	1	2	3	4
$y =$	0	2	4	6	8

Figure: Graph of $y = 2x$



Definition 3 In terms of the correspondence of two sets (Fig. 1)

A function is a correspondence between a first set, say set A and a second set, say Set B such that **each** element of set A corresponds to exactly one element of set B. The set of all the elements of set A is a called the **domain** of the function, and the set of all the corresponding elements of set B is called the **range** of the function.

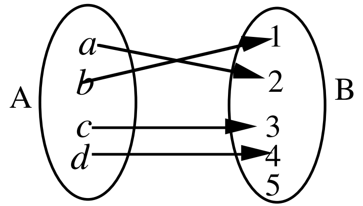


Fig 1

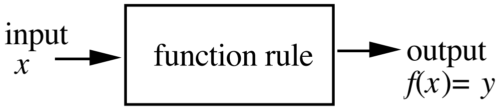


Fig 2

Definition 4 In terms of a rule for inputs and outputs (Fig. 2)

A function is a rule which assigns to each input number exactly one output number. The set of all input numbers that the rule is applicable to is called the **domain** of the function; and the set of all the corresponding output numbers is called the **range** of the function.

A variable representing an input number is called the independent variable, and a variable representing an output number is called the dependent variable.

Given a graph (Vertical line test)

A given graph is that of a function if every possible vertical line drawn to intersect the graph intersects (cuts) the graph exactly once (i.e., at one point only).

Comparison of a Function and a Relation

Similarities: Each is a set of ordered pairs.
Differences: In a relation, two or more ordered pairs may have the same first component; but in a function, no two distinct ordered pairs may have the same first component.
Example: The set $D = \{(1, 6), (3, 4), (3, 5), (4, 6)\}$ is only a relation and **not** a function. because the second and third ordered pairs have the same first component, which is 3.

Example: The set $E = \{(1, 2), (2, 3), (4, 5), (7, 5)\}$ is a function (even though the second components of the third and fourth ordered pairs are the same).

A function is a relation, but a relation is not necessarily a function.

Determining if a given graph is a relation or a function

We will use the so-called **vertical line test.**

Procedure

Step 1: Draw as many vertical lines as possible (This can be done visually.) to intersect the graph.

Step 2: If any of the possible lines intersects (cuts) the graph at more than one point, then the given graph is not a function but a relation. However, if each of the possible vertical lines intersects the graph only once (at one point only), then the graph represents a function. Figure.. is a graph which is a relation but not a function.

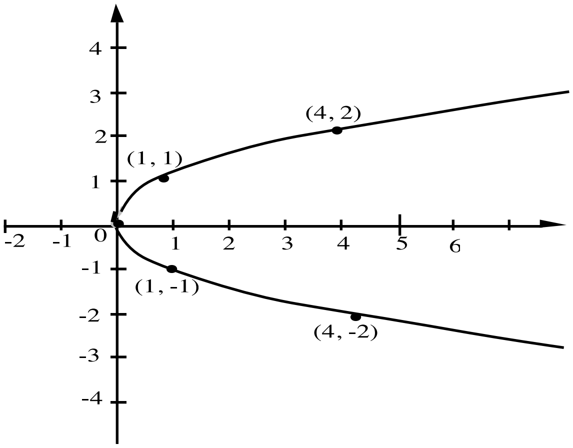


Figure: Graph of $y = \pm\sqrt{x}$ or $x = y^2$
This graph is a relation but not a function

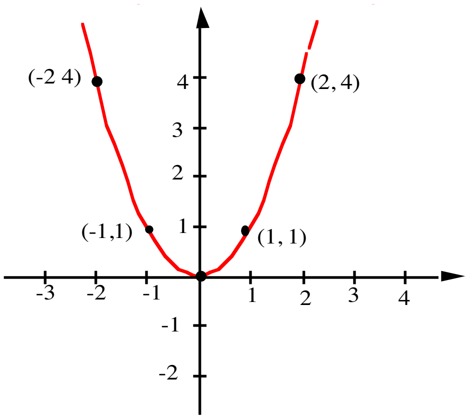


Figure: Graph of $y = x^2$.
This graph is that of a function.

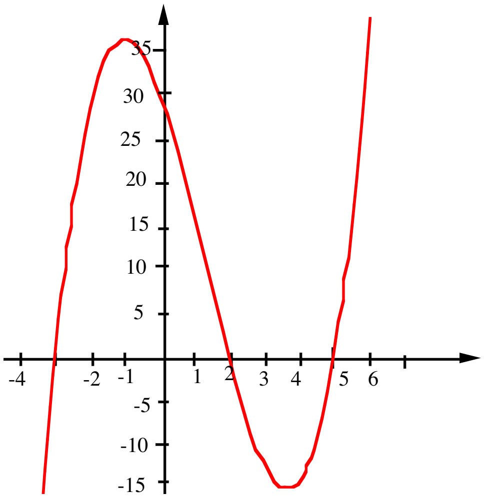


Figure: Graph of $y = (x - 5)(x - 2)(x + 3)$. This graph is that of a function.

Lesson 9 Exercises

Determine which of the following are graphs of functions.

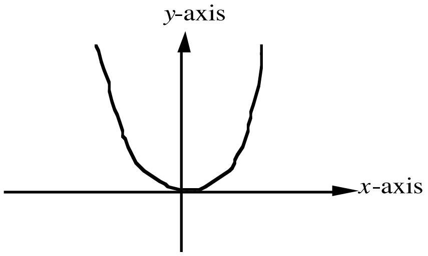


Figure (a)

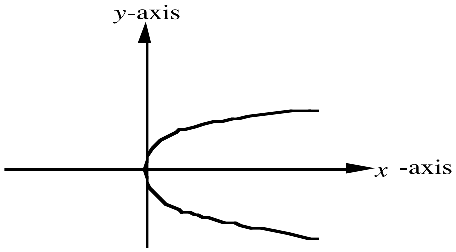


Figure (b)

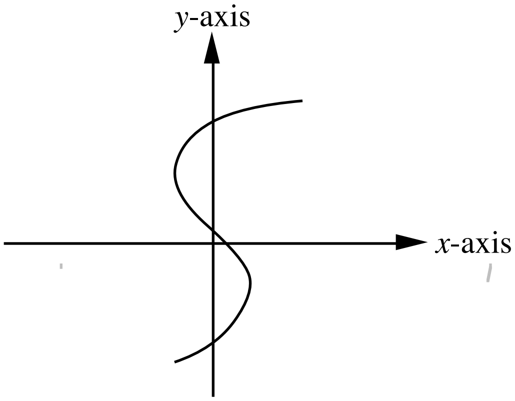


Figure (c)

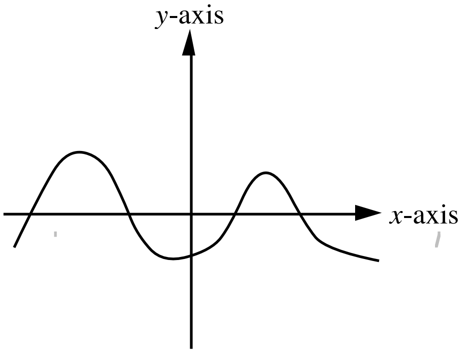


Figure (d)

Answers: (a) A function; (b) Not a function; (c) Not a function; (d) A function

Lesson 10A

Functional Notation; Defined Functions; Excluded Values; Domain and Range

Functional Notation

Let a function $f(x)$ be specified by the rule $f(x) = x^2 + 3$. (1)
In equation (1), $f(x)$ is read "*f* of *x*" or *f* is a function of *x*.

To evaluate a function for a particular value of *x*, we substitute that value of *x* in the rule that defines $f(x)$. Note that $f(x)$ does not mean *f* times *x* but that $f(x)$ is written as a symbol.

Example 1 Given that $f(x)= x^2 + 3$, find (a) *f*(-1).; (b) $f(x_0 + h) - f(x_0)$

Solution: (a) $f(x) = x^2 + 3$

$$f(-1) = (-1)^2 + 3 \qquad \text{(replacing } x \text{ in the given equation by -1)}$$
$$= 1 + 3$$
$$f(-1) = 4$$

(b) (Replace *x* in the given equation by $(x_0 + h)$ and x_0, accordingly in $f(x)= x^2 + 3$

$$f(x_0 + h) - f(x_0) = [(x_0 + h)^2 + 3] - (x_0{}^2 + 3)$$
$$= [x_0{}^2 + 2hx_0 + h^2 + 3] - x_0{}^2 - 3$$
$$= x_0{}^2 + 2hx_0 + h^2 + 3 - x_0{}^2 - 3$$
$$= 2hx_0 + h^2$$

Example 2 Find *f*(2), given that $f(x) = x + 7$

Solution

$$f(x) = x + 7$$
$$f(2) = 2 + 7$$
$$= 9$$

Example 3 If $f(x) = 2 - \frac{1}{x - 4}$, find (*a*) *f*(-3); (b) $f(x_0 + h)$; (*b*) $f(-x)$.

Solution (*a*) $f(x) = 2 - \frac{1}{x - 4}$

$$f(-3) = 2 - \frac{1}{(-3) -4} \qquad \text{(replacing } x \text{ in the given equation by -3)}$$
$$= 2 - \frac{1}{-7}$$
$$= 2 + \frac{1}{7}$$
$$= 2\tfrac{1}{7}$$

(c) (Replace *x* in the given equation by $x_0 + h$

$$f(x) = 2 - \frac{1}{x-4} \text{ <- - - - given equation}$$

$$f(x_0 + h) = 2 - \frac{1}{(x_0 + h) - 4} \text{ <- - -replacing } x \text{ by } x_0 + h$$

$$= 2 - \frac{1}{x_0 + h - 4}$$

$$= \frac{2(x_0 + h - 4) - 1}{x_0 + h - 4}$$

$$= \frac{2x_0 + 2h - 8 - 1}{x_0 + h - 4}$$

$$= \frac{2x_0 + 2h - 9}{x_0 + h - 4}$$

(*c*) (Replace *x* in the given equation by -*x*)

$$f(x) = 2 - \frac{1}{x-4} \qquad \text{<- - - - given equation}$$

$$f(-x) = 2 - \frac{1}{(-x) - 4} \qquad \text{<- - - - -}(\textit{replacing x by} - x)$$

$$= 2 - \frac{1}{-x-4}$$

$$= \frac{2(-x-4) - 1}{-x-4}$$

$$= \frac{-2x - 8 - 1}{-x-4}$$

$$= \frac{-2x - 9}{-x-4}$$

$$= \frac{-(2x+9)}{-(x+4)} \qquad (\textit{factoring out} - 1)$$

$$= \frac{2x+9}{x+4}$$

Note above that in (*b*) the final result contains *x*. This is so, because we replaced *x* by -*x*. In the case of (*a*), we replaced *x* by the integer -3 , and the final result was purely a numerical value. Furthermore, in Example 3, $f(-a) = \dfrac{2a+9}{a+4}$

Defined Real-Valued Function

Meaning of a defined function of *x*

A real-valued function $f(x)$ is said to be defined for a variable x if the following conditions are satisfied:

1. The x and $f(x)$ must be real (i.e., x and $f(x)$ should not be the square root or an even root of a negative number). Thus, a value such as $\sqrt{-4}$ or $2i$ is not allowed.

For example, in $f(x) = \sqrt{x - 4}$, we have to make sure that $x \geq 4$, since otherwise, we obtain imaginary numbers.

2. The value of x when substituted in the functional equation should yield specific real numbers. (i.e., the value of x when substituted in the functional equation should **not** make the denominator become zero.)

Condition (2) implies that the function should not become undefined when the value of x is substituted in the functional equation. When the function involves a denominator, we have to make sure that the denominator is not allowed to be zero. A particular example of this function occurs when the given function is the ratio of two polynomial functions. (We call such functions rational functions.)
In this book, it is agreed that a function is real-valued unless otherwise specified.

Examples of rational functions are: (a) $f(x) = \dfrac{x^2 + 4}{x - 1}$; (b) $f(x) = \dfrac{1}{(x - 3)(x + 4)}$

Excluded Values, Domain and Range

The **excluded values** are (usually) the values which when substituted in the functional equation make the function either undefined or imaginary.

The **domain** (say , D) of a function $f(x)$ consists of those real values of the independent variable say, x, for which $f(x)$ is real and defined.

The **range** (say, R) of the function consists of the corresponding values which $f(x)$ assumes for x in the domain of the function .

Specification of Domain and Range

The domain and the range of a given function may be specified in several ways, namely in the form of a set, in the form of a table, in the form of an equation, in the form of an inequality, or in the form of a graph.

(In some of the examples that follow, we will use a graphing calculator or a computer grapher to generate graphs which will be useful in determining the range of some of the functions. Determining the range of some of the functions analytically may require advanced methods (which we do not cover in this book) , and therefore, we use graphing as an aid.)

Example 1: In the form of a set

Find the domain and range of the function specified by $\{(1, 2), (3, 2), (4, 4), (5, 5)\}$.

Solution: The domain consists of the first components of the ordered pairs.
The domain, $D = \{1, 3, 4, 5\}$.
The range consists of the second components of the ordered pairs.
The range, $R = \{2, 4, 5\}$.

Example 2: In the form of a table of values

Find the domain and range of the function specified by the table of values below.

x	y
1	2
3	2
4	4
5	5

Solution: The domain consists of (the x-values) 1, 3, 4, and 5.

The range consists of the (y-values) 2, 4 and 5.

Note that this form is the set form written in a different format.

Example 3: In the form of an equation:

The domain consists of those real values of x for which the function is defined. The corresponding values of $f(x)$ form the range of the function.

Example 4: In the form of a graph

Find the domain and range of the function specified by the graph below.

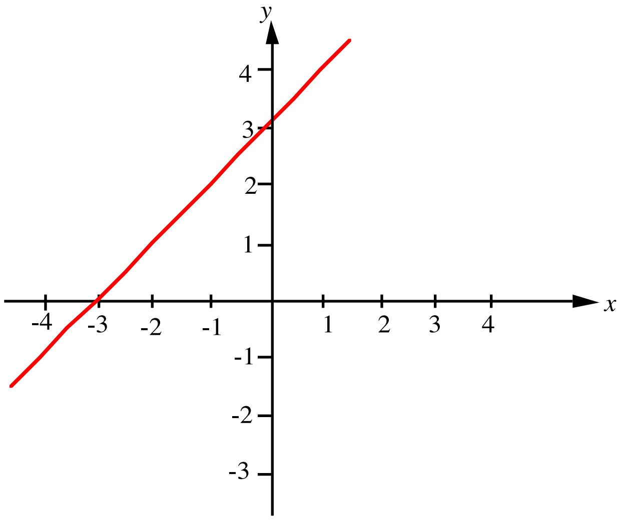


Figure: The graph of $y = x + 3$

Solution: From the graph, the function is defined for all real values of x.

The domain consists of all real x-values. The range consists of all real y -values.

Implicit and Explicit Specification of the Domain of a Function

The domain of a function may be specified either implicitly or explicitly.

Consider the function $f(x) = x^2$

As it stands, the domain is implicitly specified. The above function is that of a polynomial and as such the domain consists of all real numbers. Here, we assume the largest possible domain.

Now, consider $f(x) = x^2 \qquad 0 \le x \le 5$

In this case, the inequality written to the right of the function specifies explicitly and restricts the domain. The domain of this function is such that x is between 0 and 5, including 0 and 5.

If the inequality to the right had not been indicated, the domain would have consisted of all real x-values.

The restrictions on the domains are very important and useful in sketching the graphs of functions.

Other functions with explicitly specified domains are

$(a)\ \ y = \sin x \qquad 0 \le x \le 2\pi$

$(b)\ \ y = \cos x \qquad -2\pi \le x \le 2\pi$

Determining the Excluded Values, Domain and Range of a Function

Case 1: Polynomial functions

Example 1 (a) For what values of x is the following function not defined? (b) what is the domain?

(c) what is the range? $f(x) = x^2 - 3x + 1$

Solution: All polynomial functions are defined for all real values of the independent variable.

(a) Since the given function is a polynomial function, it is defined for all real values of x. We may note that the right-hand side of the equation does not involve denominators or square roots (or even roots) of the independent variable. There are **no** excluded values.

(b) The domain consists of all real values of x. Set-builder notation: Domain = $\{x \mid x \text{ is a real number}\}$
Using interval notation: Domain =(- ∞, +∞)

(c) Generally, determining the range of a polynomial function may require advanced methods. However, since the given function is a quadratic function , we may apply the range inequality formula, $y \geq \frac{4ac - b^2}{4a}$

From the function, $a = 1$, $b = -3$, and $c = 1$.
Substituting these values,

$$y \geq \frac{4(1)(1) - (-3)^2}{4(1)} = -\frac{5}{4} \quad (\text{i.e., } y \geq -\tfrac{5}{4})$$

Set-builder: $\{y \mid y \geq -\frac{5}{4}\}$

The range consists of all real numbers greater than or equal to $-\frac{5}{4}$.

(Note that any horizontal line drawn below the point $(\frac{3}{2}, -\frac{5}{4})$ will **not** intersect the curve, but any horizontal line through or above this point will intersect the curve)

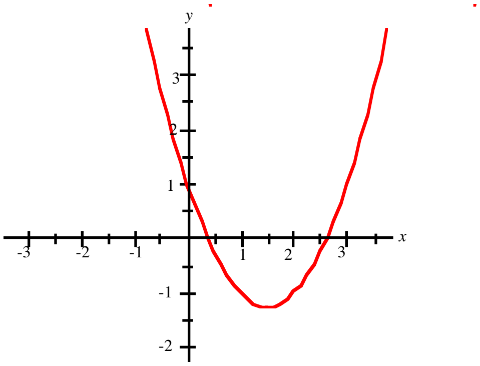


Figure: Graph of $f(x) = x^2 - 3x + 1$

Case 2: **Rational functions**

Note: A rational function is a function which is the ratio of two polynomial functions.

Example 2 (a) For what values of x is the following function not defined? (b) what is the domain? (c) what is the range?

$$f(x) = \frac{3x - 2}{x - 1}$$

Solution Step 1: Setting the denominator to zero,

$$x - 1 = 0$$

Step 2: Solving for x, $x = 1$.

(a) The function is not defined when $x = 1$. (The excluded value of x is 1.)

(b) The domain is all real values of x, except 1. same as $\{x \mid x \text{ is a real number and } x \neq 1\}$

(c) The range (from graph) if found by being guided by the horizontal asymptote, $y = 3$. (see p.273)
The range is given by the set $\{y \mid y < 3 \text{ or } y > 3\}$ or simply $\{y \mid y \neq 3\}$
(Note that a horizontal line drawn through (0, 3) will **not** intersect the curve)

Checking for $x = 1$, $f(1) = \frac{3(1) - 2}{1 - 1} = \frac{3 - 2}{0} = \frac{1}{0}$ which is undefined .(The right-hand side is division by zero).

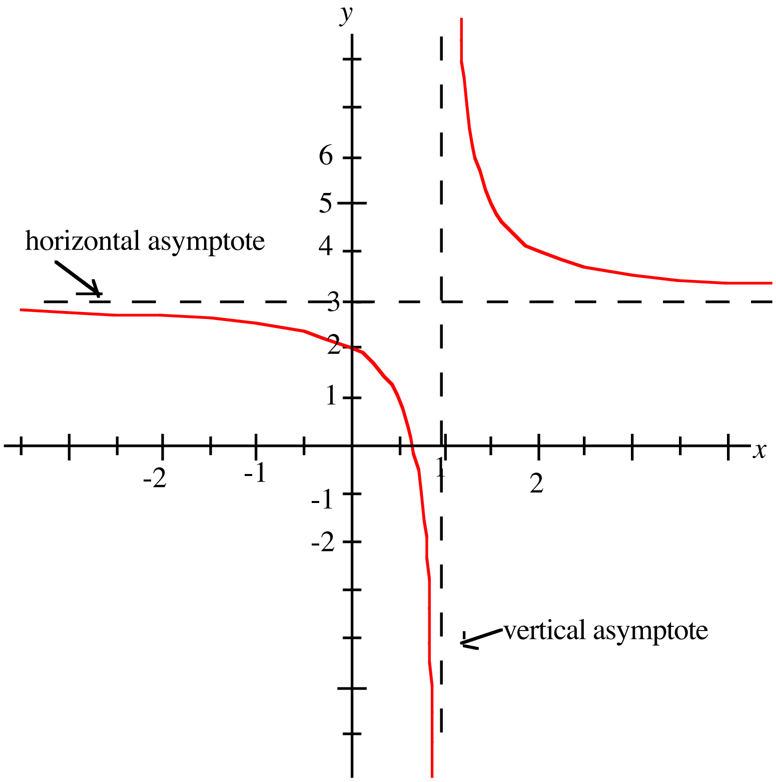


Figure: Graph of $f(x) = \frac{3x - 2}{x - 1}$

Example 3 (a) For what values of x is the given function not defined? (b) what is the domain? (c) what is the range?

$$f(x) = \frac{2(x - 1)}{(x - 2)(x + 4)}$$

Solution Step 1: Setting the denominator to zero,

$$(x - 2)(x + 4) = 0$$

Step 2: Solving for x, $x = 2$, or $x = -4$.

(a) The function is not defined when $x = 2$ and -4. (The excluded values of x are 2 and -4.)

(b) The domain is all real values of x, except 2 and -4.

Set-builder notation: $\{x \mid x$ is a real number and $x \neq -4$, $x \neq 2$ $\}$

(c) The range (from graph) is all real y.

Set-builder notation: $\{y \mid$ y is a real number$\}$

(Note that any horizontal line drawn through the y-axis will intersect the curve)

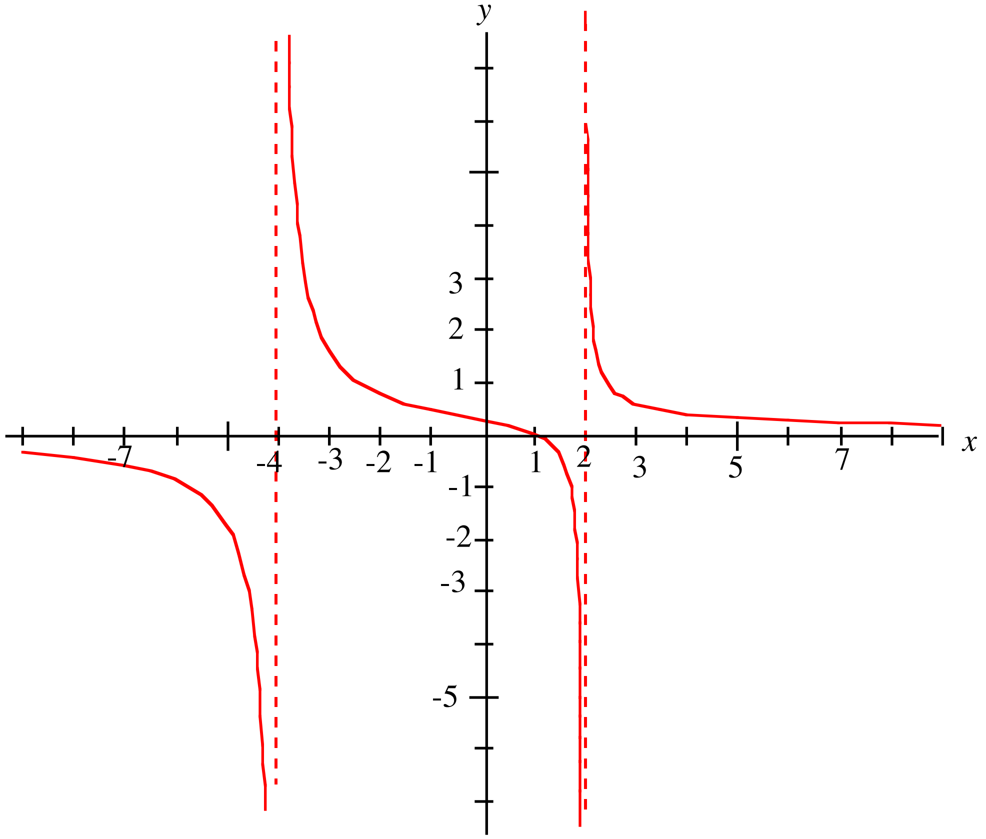


Figure: Graph of $f(x) = \frac{2(x - 1)}{(x - 2)(x + 4)}$

Example 4 (a) For what values of x is the given function not defined? (b) what is the domain? (c) what is the range?

$$f(x) = \frac{1}{x}$$

Solution Setting the denominator to zero,
$x = 0$

(a) The function is not defined when $x = 0$. (The excluded value of x is 0.)

(b) The domain is all real values of x, except 0.
Set-builder notation: $\{x \mid x \text{ is a real number and } x \neq 0\}$

(c) The range (from graph) is given by the set $\{y \mid y < 0 \text{ or } y > 0\}$ or simply $\{y \mid y \neq 0\}$
(the horizontal asymptote is $y = 0$)
(Note that any horizontal line drawn through the y-axis, except through the point $(0, 0)$, will intersect the curve.

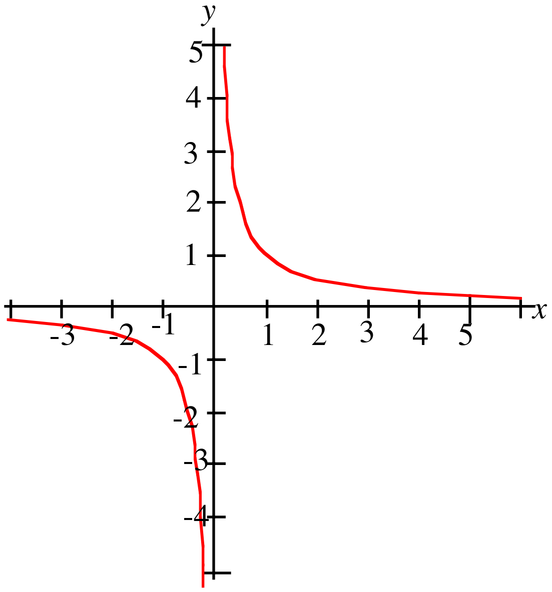


Figure: Graph of $f(x) = \frac{1}{x}$

Example 5 (a) For what values of x is the given function not defined? (b) what is the domain? (c) what is the range?

$$f(x) = \frac{x^3 + x^2 + 2}{x^2 - 16}$$

Solution: Setting the denominator to zero and solving,

$$x^2 - 16 = 0$$
$$(x + 4)(x - 4) = 0$$
$$x = -4 \text{ or } x = 4$$

The function is not defined when $x = -4$ or 4. (The excluded values of x are -4 and 4.)

(a) The domain is all real x except -4 and 4.
$\{x \mid x$ is a real number and $x \neq -4$, $x \neq 4\}$

(b) The range (from graph) is all real values of y. same as $\{y \mid y$ is a real number$\}$
(Note that any horizontal line drawn through the y-axis will intersect the curve)

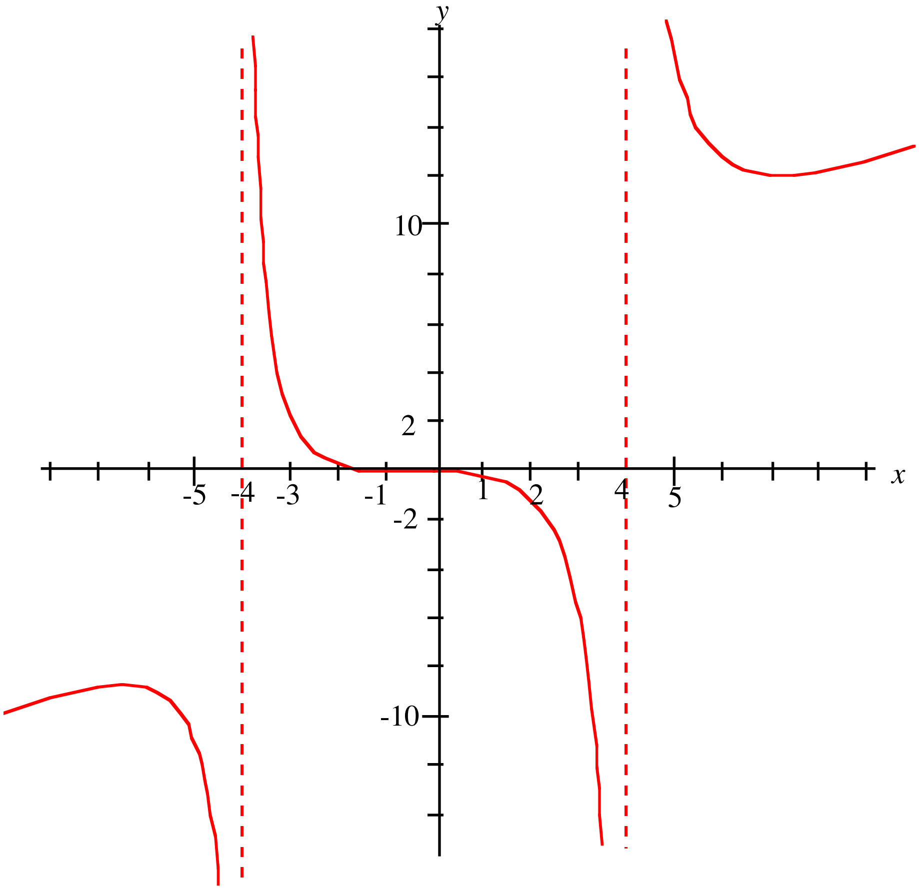


Figure: Graph of $\frac{x^3 + x^2 + 2}{x^2 - 16}$

Example 6 (a) For what values of x is the given function not defined? (b) what is the domain? (c) what is the range?

$$f(x) = \frac{8}{x^2 - 4}$$

Solution Step 1: Setting the denominator to zero,

$$x^2 - 4 = 0$$

$$(x+2)(x-2) = 0.$$

Step 2: Solving for x, $x = 2$, or $x = -2$.

(a) The function is not defined when $x = 2$ and -2. (The excluded values of x are 2 and -2.)

(b) The domain is all real values of x except 2 and -2.

Same as $\{x \mid x$ is a real number and $x \neq -2$, $x \neq 2\}$

(c) The range (from graph) is given by the set $\{y \mid y \leq -2 \text{ or } y > 0\}$

(Note that any horizontal line drawn through or below $y = -2$ or above $y = 0$ will intersect the curve)

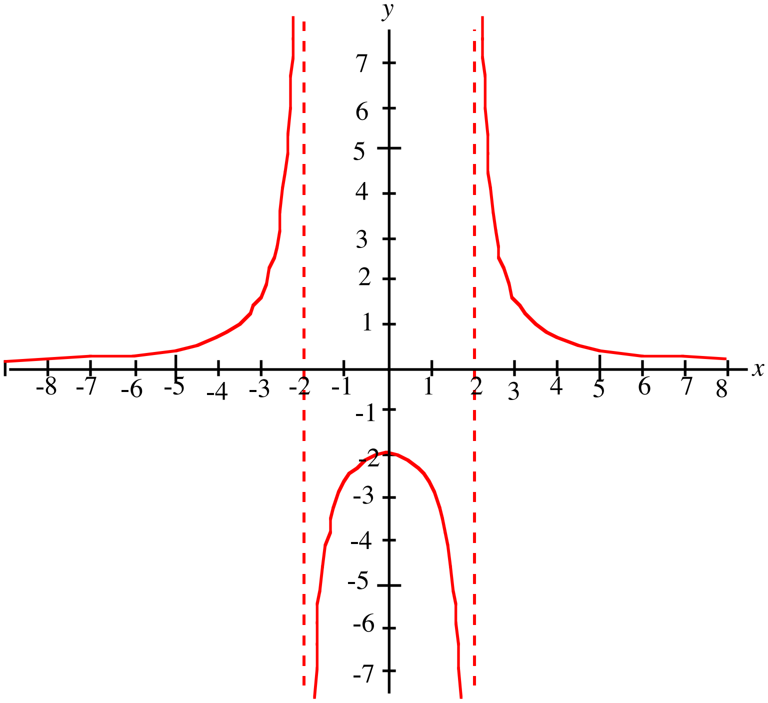


Figure: Graph of $f(x) = \frac{8}{x^2 - 4}$

Example 7 (a) For what values of x is the given function not defined? (b) what is the domain?
(c) what is the range?

$$f(x) = \frac{8}{x^2 + 4}$$

Solution Step 1: Setting the denominator to zero, and solving, we obtain non-real values. Since we are dealing with real-valued functions, we conclude that

Step 2: (a) There are **no** excluded values.
$x^2 + 4$ is positive for all real values of x and never zero, since the square of any nonzero real number is always positive.

(b) The function is defined for all real values of x.
Domain: = $\{x \mid x \text{ is a real number}\}$

(c) The range (from graph) is given by the set $\{y \mid 0 < y \leq 2\}$

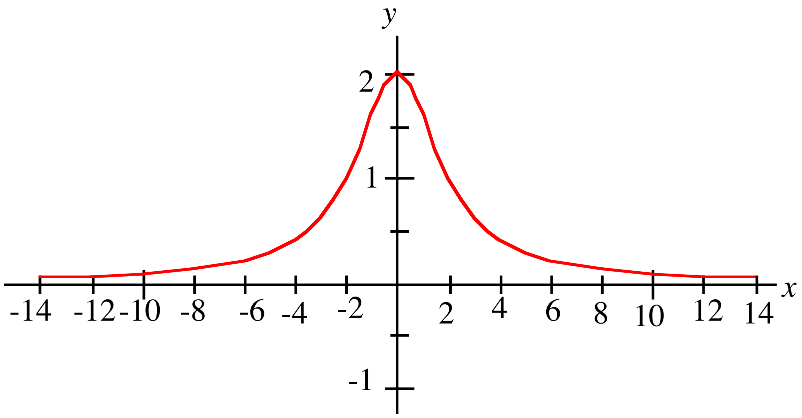


Figure: Graph of $f(x) = \frac{8}{x^2 + 4}$

Case 3: Functions containing even roots (e.g., square root) of polynomials

Example 8 (a) For what values of x is the function $f(x) = \sqrt{x - 5}$ not real?
(b) What is the domain? What is the range?

Solution (a) For real roots, the radicand, $(x - 5)$ must be positive or zero.

Symbolically, $x - 5 \geq 0$
$x \geq 5$ (solving for x).

The function is not real when $x < 5$ (because the square root would be that of a negative number. The square root of a negative number is imaginary. Any x-value less than 5 is excluded.

Checking for some specific values of x:

1. When $x = 5$, $f(5) = \sqrt{x - 5} = \sqrt{5 - 5} = 0$, which is real. (i.e., for $x = 5$).
2. When $x = 6$, $f(6) = \sqrt{6 - 5} = 1$, which is real. (i.e., for $x > 5$).
3. When $x = 4$, $f(4) = \sqrt{4 - 5} = \sqrt{-1}$, which is imaginary (non-real (i.e., for $x < 5$).

(b) The domain consists of all real values of x such that $x \geq 5$.

(c) The range for $y = \sqrt{x - 5}$ is from $y = 0$ to $y = \infty$.
(The range is obtained by considering $x \geq 5$ ($x = 5$ and $x > 5$) in $y = \sqrt{x - 5}$, or from graph (next page Fig.2)

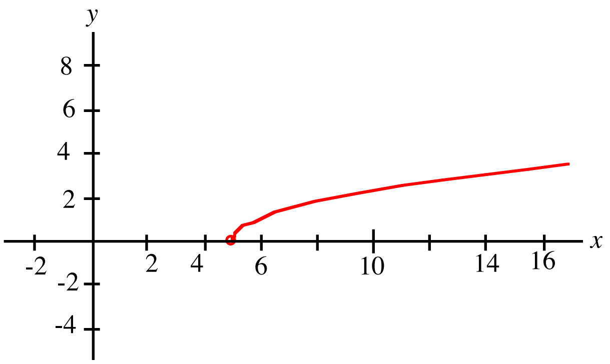


Figure 2: Graph of $f(x) = \sqrt{x - 5}$

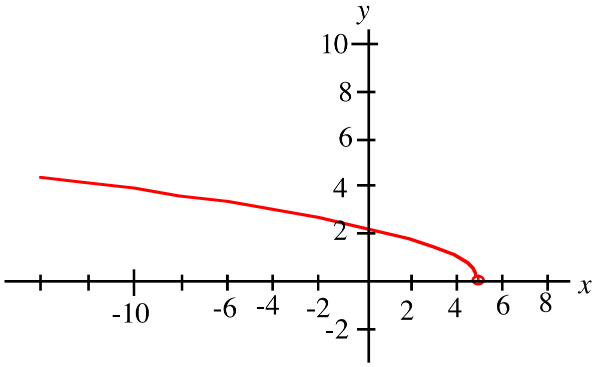


Figure 3: Graph of $f(x) = \sqrt{5 - x}$

Example 9 (a) For what values of x is $f(x) = \sqrt{5 - x})$ not real?

(b) What is the domain? What is the range?

Solution

(a) $5 - x \geq 0$ (for real values of $f(x)$, the radicand must be positive or zero).

$x \leq 5$ (solving for x)

The function is not real if $x > 5$. (Any x-value greater than 5 is excluded).

(b) The domain is all real x such that $x \leq 5$.

(c) The range is from $y = 0$ to $y = \infty$ (obtained by considering $x \leq 5$ ($x = 5$ and $x < 5$) in $y = \sqrt{5 - x}$., or (from graph)

Example 10 Find (a) the domain and (b) the range of $f(x) = \dfrac{\sqrt{x + 6}}{x - 1}$

Solution Here, two conditions must be satisfied.

Condition 1: $x + 6 \geq 0$ and from which $x \geq -6$ (For real root, the radicand must be positive or zero)

Condition 2: $x - 1 \neq 0$ and from which $x \neq 1$.

(a) The domain consists of all real numbers greater than or equal to -6, except 1, that is $x \geq -6, x \neq 1$.

(b) The range (from graph) is given by the set $\{y \mid y \text{ is a real number}\}$. Note that when $x = -6$, $y = 0$

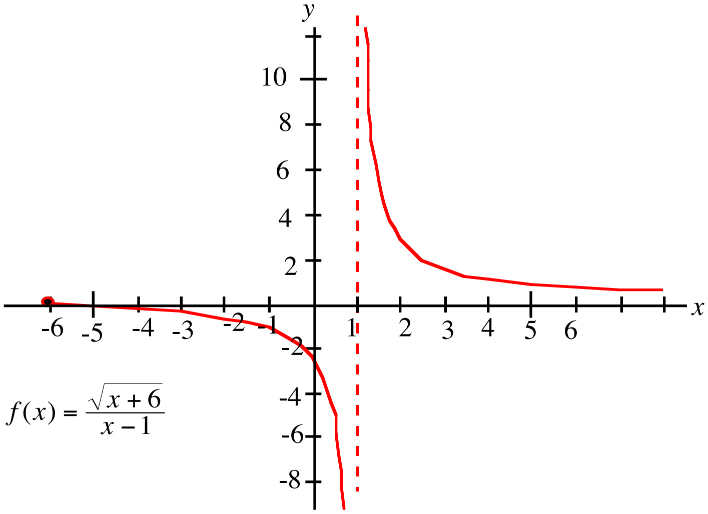


Fig.4: $f(x) = \dfrac{\sqrt{x + 6}}{x - 1}$

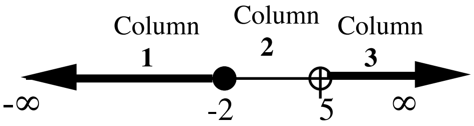
Example 11 Determine (a) the excluded values of x ; (b) the domain ; (c) the range of

$$f(x) = \sqrt{\frac{x+2}{x-5}}$$

Solution

For real and defined values of $\sqrt{\frac{x+2}{x-5}}$, $\frac{x+2}{x-5} \geq 0$ ($x+2 \geq 0$ and $x-5>0$ or $x+2 \leq 0$ and $x-5<0$)

We will use a sign diagrams to solve this problem. (See also chapter 12)



	Factor	Signs of the intervals		
Row 1	$x+2$	–	+	+
Row 2	$x - 5$	–	–	+
Row 3	$\frac{x+2}{x-5}$	+	–	+

In Row 3, Columns 1 and 3 (with the "+" signs), we read that $\frac{x+2}{x-5} \geq 0$ if $x \leq -2$ or $x > 5$

(a) For these columns,, the excluded values are such that $-2 < x \leq 5$.

(b) The domain is such that $x \leq -2$ or $x > 5$

(c) The range (from graph) is given by $\{y \mid 0 \leq y < 1 \text{ or } y > 1\}$.

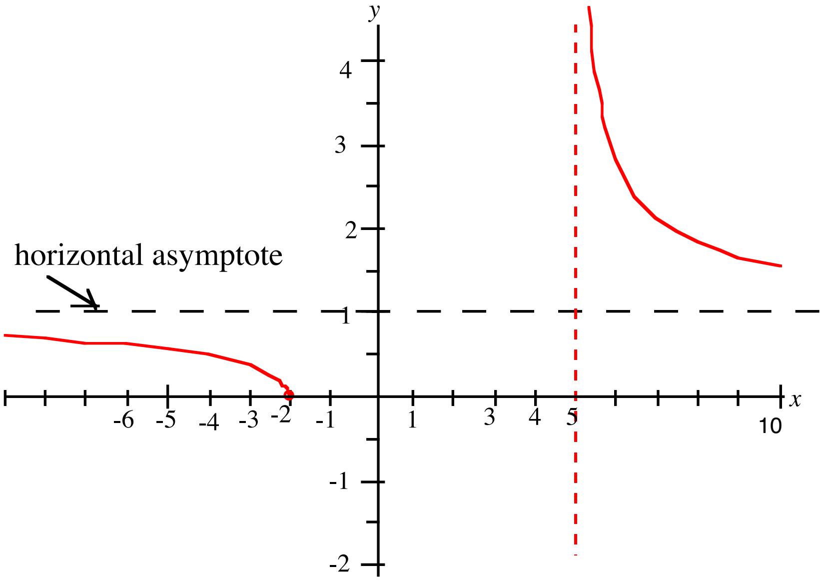


Figure: Graph of $f(x) = \sqrt{\frac{x+2}{x-5}}$

Lesson 10A Exercises

1. If $(x) = x^2 - 5x + 2$, find $(a)\ f(-2)$; $(b)\ f(-1)$, $(c)\ f(-x)$, $(d)\ f(a+h)$, $(e)\ \dfrac{f(a+h) - f(a)}{h}$;

A: **2**. If $(x) = 4 - \dfrac{1}{x-2}$, find $(a)\ f(-3)$; $(b)\ f(-x)$; $(c)\ f(a)$.

3. If $(x) = x^3 - x^2 - x - 1$, find $f(-1)$.

Answers: **1**. (a) 16; (b) 8; (c) $x^2 + 5x + 2$; (d) $a^2 - 5a + 2ah - 5h + h^2 + 2$**;** (e) $2a + h - 5$

2. (a) $4\frac{1}{5}$; (b) $\dfrac{4x+9}{x+2}$; (c) $\dfrac{4a-9}{a-2}$; **(3)** -2

B **1**. Determine the domain of $f(x) = \dfrac{\sqrt{x+2}}{\sqrt{x-5}}$.

Hint: .The following conditions must be satisfied simultaneously:

a. $x + 2 \geq 0$ or $x \geq -2$ (so that the square root is not imaginary)

b. $x - 5 \geq 0$ or $x \geq 5$ (so that the square root is not imaginary)

c $x - 5 \neq 0$ or $x \neq 5$ (Since otherwise the function is undefined)

1. Domain: $x > 5$**.** (see the graph in Example 11 above for $x > 5$)

Note above: $\dfrac{\sqrt{x+2}}{\sqrt{x-5}} \neq \sqrt{\dfrac{x+2}{x-5}}$ (For real numbers a and b, $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$ if $a \geq 0$ and $b > 0$)

C **1**.What is meant by the domain of a function?

2.What conditions do you look out for in determining the domain of a function?

3.What is meant by the range of a function?

Find the domain and range in each of the following:

4. $A = \{(2,4), (3,9), (4,16), (5,25)\}$; **5**. $B = (2,1),\ (5,1),\ (6,1),\ (7,3)$

6. $y = x + 2$; **7**. $y = x^2 + 3x + 7$; **8**. $y = -x - 4$; **9**. $y = x^3 + x^2 - 5$

Answers: **4**. Domain : $\{2, 3, 4, 5\}$;, range: $\{4, 9, 16, 25\}$

5. Domain : $\{2, 5, 6, 7\}$;, range: $\{1,\ 3\}$

6. Domain : All real values of x: range: all real values of y.

7. Domain : All real x; range: all real y such that $y \geq 4.75$**.**

8. Domain : All real values of x: range: all real values of y.

9. Domain : All real values of x.

D Determine the excluded values (if any) for the following:

1. $y = \dfrac{2}{x-3}$; **2**. $f(x) = \dfrac{2x-4}{x^2-1}$; **3**. $f(x) = \dfrac{3x}{x^2+1}$; **4**. $f(x) = \dfrac{1}{x}$; **5**. $f(x) = \dfrac{(x-1)}{(x-1)(x+2)}$

In Problems 6 & 7 below, for what values of x is the function not real?

6. $y = \sqrt{x-3}$; **7**. $y = \sqrt{4-x}$ **8**. Find the domain of $f(x) = \sqrt{\dfrac{x+2}{x-5}}$.

9. Find the domain of $f(x) = x - \dfrac{1}{x}$

Extra: Why is it important to note the excluded values in dealing with functions and solving equations?

Answers: **1**. 3; **2**. {-1, 1}; **3**. None; **4**. 0; **5**. {-2, 1}; **6**. $x<3$; **7**. $x>4$; **8**. $x \leq -2$ or $x > 5$

9. All real x except 0.

Lesson 10B
Algebra of Functions

In the past, we covered the addition, subtraction, multiplication and division of algebraic expressions. Below, we similarly cover the same basic operations on functions.

Let *f* and *g* be functions of *x* *such* that *x* is in the domains of both *f* and *g*. Then

1. Sum: $(f+g)(x) = f(x) + g(x)$ (Domain: All real numbers common to the domains of *f* and *g*)

2. Difference: $(f-g)(x) = f(x) - g(x)$Domain: All real numbers common to the domains *f* and *g*)

3. Product: $(f \bullet g)(x)$ or $(fg)(x) = f(x) \bullet g(x)$ or $f(x)g(x)$ (Domain: All real numbers common to the domains *f* and *g*

4. Quotient $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ (Domain: All real numbers common to the domains *f* and *g* and $g(x) \neq 0$.

Examples

If $f(x) = x^2 + 2$ *and* $g(x) = x - 2$; find (a) $(f+g)(x)$; (b) $(f-g)(x)$; (c) $(f \bullet g)(x)$; (d) $\left(\frac{f}{g}\right)(x)$; (e) $(f+g)(3)$; (f) $\left(\frac{f}{g}\right)(3)$; (g) $\left(\frac{f}{g}\right)(2)$.

Solution

(a) $(f+g)(x) = f(x) + g(x)$

$$= (x^2+2) + (x-2)$$
$$= x^2 + 2 + x - 2$$
$$= x^2 + x$$

(b) $(f-g)(x) = f(x) - g(x)$

$$= (x^2+2) - (x-2)$$
$$= x^2 + 2 - x + 2$$
$$= x^2 - x + 4$$

(c) $(f \bullet g)(x) = f(x) \bullet g(x)$

$$= (x^2+2)(x-2)$$
$$= x^3 - 2x^2 + 2x - 4$$

(d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$= \frac{x^2+2}{x-2} \text{ or } x + 2 + \frac{6}{x-2}$$ (long division)

(e) From (a), $(f+g)(x) = f(x) + g(x)$

$$= x^2 + x$$
$$(f+g)(3) = (3)^2 + 3 = 12$$

Method 2: $(f+g)(3) = f(3) + g(3)$

$$f(3) = (3)^2 + 2 = 11$$
$$g(3) = 3 - 2 = 1$$
$$(f+g)(3) = 11 + 1 = 12$$

(f) From (d), $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$= \frac{x^2+2}{x-2}$$
$$\left(\frac{f}{g}\right)(3) = \frac{(3)^2+2}{(3)-2}$$
$$= \frac{9+2}{3-2}$$
$$= 11.$$

Method 2: $f(3) = (3)^2 + 2 = 11$

$$g(3) = 3 - 2 = 1$$
$$\left(\frac{f}{g}\right)(3) = \frac{f(3)}{g(3)} = \frac{11}{1} = 11$$

(g) From (d), $\left(\frac{f}{g}\right)(x) = \frac{x^2+2}{x-2}$

$$\left(\frac{f}{g}\right)(2) = \frac{(2)^2+2}{(2)-2}$$
$$= \frac{6}{0} \text{ is undefined.}$$

Therefore, $\left(\frac{f}{g}\right)(2)$ is undefined. Note: $g(2) = 0$.

On the next page, we cover the domains for algebraic of functions.

Domains for Algebra of Functions

In the first step, we determine the common domain (intersection of domains) of the functions. In the second step, we check for the domain of the resulting function. (sum. difference, product or quotient). We add any excluded values to those from Step 1 and if there is an overlap in the domains, we determine the intersection of the domains from Step 1 and Step 2. Pay attention to radical functions such as $f(x) = \sqrt{x-2}$, determine the domain and find the intersection for the overall domain.

Example

Given $f(x) = \frac{3}{x}$ and $g(x) = \frac{3x-4}{x+2}$, find the domains of the following:

(a) $(f+g)(x)$; (b) $(f-g)(x)$; (c) $(f \bullet g)(x)$; (d) $\left(\frac{f}{g}\right)(x)$;

Solution:

Domain of $f: \{x \mid x \text{ is a real number and } x \neq 0\}$

Domain of $g: \{x \mid x \text{ is a real number and } x \neq -2\}$.

For **(a), (b), and (c):**

domain of $f+g$ = domain of $f-g$ =domain of $f \bullet g$= $\{x \mid x \text{ is a real number and } x \neq 0,\ x \neq -2\}$

(d) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$= \frac{3}{x} \div \frac{3x-4}{x+2} \qquad (x \neq 0,\ x \neq -2)$$

$$= \frac{3}{x} \bullet \frac{x+2}{3x-4}$$

$(x \neq \frac{4}{3}$. See $= \frac{3(x+2)}{x(3x-4)}$ below)

Note: $\frac{f(x)}{g(x)} = \frac{\frac{3}{x}}{\frac{3x-4}{x+2}}$

Setting $x(3x-4) = 0$

$x = 0$ or $3x - 4 = 0$ and from which $x = \frac{4}{3}$

The excluded values are -2, 0 and $\frac{4}{3}$.

Domain of $\left(\frac{f}{g}\right)(x)$: $\left\{x \mid x \text{ is a real number and } x \neq 0,\ x \neq -2 \text{ and } x \neq \frac{4}{3}\right\}$

Extra

Example: The intersection of the domains of $f(x) = \sqrt{x-2}$ and $g(x) = \frac{x-4}{x+3}$ is $\{x \mid x \text{ is a real number and } x \geq 2\}$ or $[2, \infty)$ (from $\{x \mid x \geq 2\} \cap \{x \mid x \neq -3\}$)

It seems the radical function is "King".

Lesson 10B Exercises

If $f(x) = x^2 + 2$ *and* $g(x) = x - 2$; find (a) $(f+g)(x)$; (b) $(f-g)(x)$; (c) $(f \bullet g)(x)$; (d) $\left(\frac{f}{g}\right)(x)$; (e) $(f+g)(3)$; (f) $\left(\frac{f}{g}\right)(3)$; (g) $\left(\frac{f}{g}\right)(2)$; (h) $(f-g)(4)$.

Ans: (a) $x^2 + x$; (b) $x^2 - x + 4$; (c) $= x^3 - 2x^2 + 2x - 4$; (d) $\frac{x^2+2}{x-2}$ or $x + 2 + \frac{6}{x-2}$; (e) 12; (f) 11 (g) undefined; (h) 16.

Lesson 11

One-to-One Functions, Composite Functions

One-to-One Functions

(In Lesson 12, we will learn that if a function is one-to-one, then it has an inverse function.)
We consider three main cases according to how the function is specified.

Case 1: **Given a set of ordered pairs** (or a table of x- and y-values)

A one-to-one (1-1) function is a set of ordered pairs in which for any two ordered pairs, the first elements are different from each other and the second elements are also different from each other.

Example The set, $A = \{(3,2),(4,7),(1,5),(2,3)\}$ is a one-to-one function. However.
The set, $B = \{(3,2),(4,7),(1,5),(5,2)\}$ is **not** a one-to-one function because the first and the last ordered pairs have the same second elements, namely, 2.

Case 2: Given the graph of the function

By the so called **horizontal line test**, the graph of a function is that of a one-to-one function if every possible horizontal line drawn to intersect the graph cuts (intersects) the graph only once (at one point only). **Figures** 2 and 3, p. 75, are graphs of one-to-one functions but **Figure 1** is not one-to-one.

Case 3: Given the equation of the function

A function $f(x)$ is one-to-one if whenever $f(x_1) = f(x_2)$, $x_1 = x_2$**.** If we let $y_1 = f(x_1)$, and $y_2 = f(x_2)$, then $y_1 = y_2$ **must** imply that $x_1 = x_2$, otherwise, $f(x)$ is not one-to-one.

Example 2: Determine if $f(x) = 2x + 3$ is one-to-one.

Solution

Step 1: $f(x_1) = 2x_1 + 3$

$f(x_2) = 2x_2 + 3$

Equate RHS of $f(x_1)$ to RHS of $f(x_2)$
(That is, let $f(x_1) = f(x_2)$):

$$2x_1 + 3 = 2x_2 + 3 \qquad (1)$$

Step 2: Solve for x_1. (You may also solve for x_2)

$$2x_1 + 3 = 2x_2 + 3$$
$$2x_1 = 2x_2$$
$$x_1 = x_2 \qquad (2)$$

Since from above, whenever $f(x_1) = f(x_2)$ (from equation (1)) $x_1 = x_2$ (from equation (2))
$f(x) = 2x + 3$ is one-to-one. (You may check by sketching its graph and using the horizontal line test)

Example 3 Determine if $f(x) = \sqrt{25 - x^2}$ is one-to-one.

Solution

Step 1: $f(x_1) = \sqrt{25 - x_1^2}$

$f(x_2) = \sqrt{25 - x_2^2}$

Equate RHS of $f(x_1)$to RHS of $f(x_2)$
(That is, let $f(x_1) = f(x_2)$.)

$$\sqrt{25 - x_1^2} = \sqrt{25 - x_2^2} \qquad (1)$$

Step 2: Solve for x_1.

$$\sqrt{25 - x_1^2} = \sqrt{25 - x_2^2}$$
$$25 - x_1^2 = 25 - x_2^2$$
$$-x_1^2 = -x_2^2$$
$$x_1^2 = x_2^2$$
$$x_1 = \pm\sqrt{x_2^2}$$
$$x_1 = +x_2 \text{ or } -x_2 \qquad (2)$$

Since from above, whenever $f(x_1) = f(x_2)$, $x_1 = -x_2$, (That is for the same y-value, x is not unique: $x_1 = +x_2$, and **also** $x_1 = -x_2$ (from equation (2))
$f(x) = \sqrt{25 - x^2}$ is **not** one-to-one. (You may check by sketching its graph and using the horizontal line test)

Example 4: Determine if $f(x) = |x - 2|$ is one-to-one.

Solution

Step 1: $f(x_1) = |x_1 - 2|$

$f(x_2) = |x_2 - 2|$

Equate RHS's of $f(x_1)$to $f(x_2)$

(That is, let $f(x_1) = f(x_2)$):

$$|x_1 - 2| = |x_2 - 2| \qquad (1)$$

Step 2: Solve for x_1:

If both $x_1 - 2$ and $x_2 - 2$ are positive,

$x_1 - 2 = x_2 - 2$ and from which $x_1 = x_2$

(same result as if both $x_1 - 2$ and $x_2 - 2$ are negative)

However, if $x_1 - 2$ is positive and $x_2 - 2$ is negative,

$$x_1 - 2 = -(x_2 - 2)$$

$$x_1 - 2 = -x_2 + 2$$

$x_1 = -x_2 + 4$. That is, $x_1 \neq x_2$

Since from above, whenever $f(x_1) = f(x_2)$,$x_1 \neq x_2$

$f(x) = |x - 2|$ is **not** one-to-one. (You may check by sketching its graph and using the horizontal line test)

Another method for determining if a function is one-to-one

The author proposes the following definition for a one-to-one function.

Definition : A function is one-to-one if its inverse relation is a function.
This definition provides another method for determining if a given function is one-to-one.

Example 5

Determine if $f(x) = 2x + 6$ is one-to-one

Solution

Given: $f(x) = 2x + 6$

Required: To determine if $f(x) = 2x + 6$ is one-to-one.

Plan: If it can be shown that $f(x) = 2x + 6$ has an inverse relation which is function, then $f(x) = 2x + 6$ is one-to-one.

Determination:

Step 1: Let $f(x) = y$ to obtain $y = 2x + 6$.

Step 2: Interchange x and y to obtain the inverse relation $x = 2y + 6$

Step 3: Solve for y.

$$2y = x - 6$$

$$y = \tfrac{1}{2}x - 3$$

Since clearly, for a given value of x there is exactly one corresponding value of y.

the inverse relation, $y = \frac{1}{2}x - 3$, is a function and therefore, $f(x) = 2x + 6$ is one-to-one.

(You may also check that $y = \frac{1}{2}x - 3$ is a function by sketching its graph and applying the vertical line test)

Example 6

Determine if $f(x) = x^2$ is one-to-one.

Solution

Given: $f(x) = x^2$

Required: To determine if $f(x) = x^2$ is one-to-one.

Plan: If it can be shown that $f(x) = x^2$ has an inverse relation which is function, then $f(x) = x^2$ is one-to-one.

Determination

Step 1: Let $f(x) = y$ to obtain $y = x^2$

Step 2: Interchange x and y to obtain the inverse relation $x = y^2$

Step 3: Solve for y.

If $y^2 = x$

$y = \pm\sqrt{x}$ (that is, $y = +\sqrt{x}$ or $y = -\sqrt{x}$

Clearly, for a given value of x there are two different corresponding y-values. Therefore, y is **not** a function of x, and the inverse relation $y^2 = x$ is **not** a function and therefore , the given function $f(x) = x^2$ is **not** one-to-one.

(You may also check that $y^2 = x$ or $y = \pm\sqrt{x}$ is **not** a function by sketching its graph and applying the vertical line test)

Composite Functions

Some authors refer to composite functions as "product functions". This alternative terminology may be misleading because a reader might be inclined to multiply the given functions. Perhaps, a better alternative terminology for a composite function is " a function within another function".

Use of Composition of Functions:
The principle of composition of functions can be used to determine algebraically if two given functions are inverses of each other.

Definition: If two functions f and g are such that the range of g is in the domain of f, then the composite function of f with g, symbolized $f \circ g$, is specified by $(f \circ g)(x) = f[g(x)]$ which is read "f of g of x". That is, the output of g becomes the input for f.

Similarly, if the range of f is in the domain of g, then the composite function of g with f, symbolized $g \circ f$ is specified by $g \circ f = g[f(x)]$ which is read "g of f of x."
That is, the output of f becomes the input for g.

We must **note** that generally, $f \circ g \neq g \circ f$ (i.e., generally, $f \circ g$ is not equal to $g \circ f$)

Example 1

If $f(x) = x^2$, and $g(x) = x + 1$, find (a) $f[g(x)]$; (b) $g[f(x)]$.

Solution

Recall that if,for example,

$$f(x) = x^2$$

then $f(3) = (3)^2$.

Similarly, (a) $f[g(x)] = f[x + 1]$

$$= [x + 1]^2$$

$$f[g(x) = x^2 + 2x + 1$$

Thus, (in the above problem) wherever there is x in the equation for $f(x) = x^2$, write (substitute) $x + 1$.

(b) $g[f(x)]$

$$= g[x^2]$$

$$= (x^2) + 1$$ (substitute x^2 for x in the equation for $g(x) = x + 1$)

$$= x^2 + 1$$

Thus, wherever there is x in the equation for $g(x) = x + 1$, we substitute x^2.

Example 2: If $f(x) = 2(x + 10)$, and $g(x) = x - 2$, find (a) $f[g(x)]$; (b) $g[f(x)]$.

Solution

(a) $f[g(x)] = 2[(x - 2) + 10$

$$= 2[x - 2 + 10]$$

$$= 2[x + 8]$$

$$f[g(x)] = 2x + 16$$

(Substituting x - 2 in the equation for $f(x)$.

(b) $g[f(x)] = 2(x + 10) - 2$

$$= 2x + 20 - 2$$

$$g[f(x)] = 2x + 18$$

(Substituting $2(x + 10)$ in the equation for $g(x)$.

Domains of Composite Functions

The approach here is similar to that for domains of algebra of functions. However, here, in the first step, we check only for the domain of the "inside" function. In the second step, we check for the domain of the resulting composite function. We add any excluded values to those from Step 1 and if there is an overlap in the domains we use the intersection of the domains from Step 1 and Step 2. Pay attention to radical functions such as $f(x) = \sqrt{x-2}$,.

Given $f(x) = \frac{3}{x}$ and $g(x) = \frac{3x-4}{x+2}$, find the domains of the following:

find (a) $f[g(x)]$; (b) $g[f(x)]$.

(a) Step 1: We check the domain of the "inside function", $g(x) = \frac{3x-4}{x+2}$

Domain of $g: \{x \mid x \text{ is a real number and } x \neq -2\}$.

Step 2: Form the composite function and check its domain.

$$f[g(x)] = \frac{3}{\frac{3x-4}{x+2}} \qquad (f(x) = \frac{3}{x})$$

$$= \frac{3}{1} \bullet \frac{x+2}{3x-4}$$

$$f[g(x)] = \frac{3(x+2)}{3x-4}$$

Set $3x-4=0$ to obtain $x = \frac{4}{3}$. Therefore $x \neq \frac{4}{3}$.

Combining the domains from Step 1 and Step 2, the excluded values are -2, and $\frac{4}{3}$.; and the domain of $f[g(x)] = \frac{3(x+2)}{3x-4}$ is

$\{x \mid x \text{ is a real number and } x \neq -2 \text{ and } x \neq \frac{4}{3}\}$

(a) Step 1: We check the domain of the "inside function", $f(x) = \frac{3}{x}$

Domain of $f: \{x \mid x \neq 0\}$.

Step 2: Form the composite function and check its domain.

$$g[f(x)] = \frac{3(\frac{3}{x})-4}{\frac{3}{x}+2} \qquad (g(x) = \frac{3x-4}{x+2})$$

$$g[f(x)] = \frac{-4x+9}{2x+3}$$

Set $2x+3=0$ to obtain $x = -\frac{3}{2}$.

Therefore $x \neq -\frac{3}{2}$.

Combining the domains from Step 1 and Step 2, the excluded values are 0, and $-\frac{3}{2}$.; and the domain of $g[f(x)] = \frac{-4x+9}{2x+3}$ is

$\{x \mid x \text{ is a real number and } x \neq -0 \text{ and } x \neq -\frac{3}{2}\}$

Application of Composition of Functions

If two functions f_1 and f_2 are inverses of each other, then, the following two conditions must be satisfied simultaneously. **1**. $f_1[f_2(x)] = x$ and **2**. $f_2[f_1(x)] = x$. For examples, see page 80.

Lesson 11 Exercises

A Show and determine which of the following functions are one-to-one

1. The set $\{(4,5), (3,4), (1,2)\}$; 2. The set $\{(2,2), (4,3), (5,7)\}$; 3. $f(x) = 3x-2$;

4. $f(x) = \sqrt{x+3}$; **5**. $f(x) = \sqrt{16-x^2}$; **6**. $f(x) = |x|$; **7**. $f(x) = |x-2|$; **8**. $f(x) = x^3 + 2$.

Answers: **1**. Yes; **2**. Yes; **3**. Yes; **4. Yes**; **5**. No; 6. No; **7**. No.; 8. Yes

B **1**. Given that $f(x) = x+2$, $g(x) = x-1$, find (a) $f[g(x)]$; (b) $g[f(x)]$

2. Given that $f(x) = 3(x+2)$, $g(x) = \frac{1}{x}$ find (a) $f[g(x)]$; (b) $g[f(x)]$

3. Given that $f(x) = (x+1)^2 + 3$, $g(x) = -x$, find (a) $f[g(x)]$; (b) $g[f(x)]$

Answers: **1**. (a) $x + 1$; (b) $x + 1$; **2**. (a) $\frac{3}{x} + 6$; (b) $\frac{1}{3x+6}$; **3**. (a) $x^2 - 2x + 4$; (b) $-x^2 - 2x - 4$

Lesson 12

Inverse Functions and Inverse Relations

(Exchange is no Robbery)

Recalling the definitions of a relation and a function (page 48), a relation is a set of ordered pairs and function is a relation in which no two distinct ordered pairs have the same first elements (components).

Inverse Relation

Given a relation which is specified by the set of ordered pairs (x, y), the inverse relation of this given relation is the set of ordered pairs (y, x). This inverse relation is obtained by interchanging the first and second elements of each ordered pair.

Example 1: Find the inverse relation of the function specified by the set A:

$$A = \{(1, 3), (3, 2), (5, 7)\}$$

Solution The inverse relation is obtained by interchanging the first and second elements of each ordered pair

The inverse relation of A is the set

$$\{(3,1),(2,3),(7,5)\}$$

From the definition of the inverse, we can conclude that when we form the inverse of a relation, the domain and the range of the original relation are interchanged. Thus, the domain of the original relation becomes the range of the inverse, and the range of the original relation becomes the domain of the inverse. The inverse may be a relation or a function. (See page 48)

Inverse function

Let a function be specified by the set of ordered pairs $\{(x, y)\}$. Then the inverse relation of this function is the set of ordered pairs $\{(y, x)\}$. If this inverse relation is also a function, then for the inverse relation ,we symbolize $f^{-1}(x)$, which is read "f inverse of x" and we say that $f(x)$ and $f^{-1}(x)$ are inverse functions of each other.

Example 2 Find the inverse function of the function specified by the set B,:

$$B = \{(a, b), (c, d), (e, f)\}$$

Solution: The inverse function of B is the set, denoted by B^{-1}, is given by

$$B^{-1} = \{(b, a), (d, c), (f, e)\}$$

Sometimes, authors ask the question :" Does this function have an inverse?" Such a question may sometimes be misunderstood, since the inverse is found by interchanging the roles of x and y, which is always possible. What is implied in such a question is whether or not the inverse relation obtained is **also** a function. Perhaps, an unambiguous form of the question should be "Does this function have an **inverse function**?" If the inverse relation of a given function is also a function, then we also say that the given function is invertible

Note that it is possible that a given relation which is not a function may have an inverse relation which is a function.

Let us elucidate how the terms, " relation, function and inverse", are connected by using the terms "inverse relation" and "inverse function" in the following statements:

Every relation has an inverse relation. This inverse relation may or may not be a function.

Every function has an inverse relation. Some functions have inverse relations which are (inverse) functions.

A function is also a relation , but a relation is not necessarily a function.

Determining if a function has an inverse function

Necessary and sufficient condition for a function to have an inverse function: A function has an inverse function if and only if it is a one-to-one function.

We will consider the different forms in which the rules for specifying a function may be given and determine if the function has an inverse function. We will consider the set form, the tabular form, the graphical form, and the equation form.

Case 1: Given the set form of the function

A function has an inverse function if for any two different ordered pairs, the second elements are different. If a function has an inverse function it is said to be invertible.

The necessary and sufficient condition is a consequence of the definitions of a function (page 48) and of an inverse function. We may note here that the condition for invertibility refers to the differences in the second components while the condition for being a function refers to the differences in the first components.

Example 3 Given the function specified by the set

$\mathrm{A} = \{(1,2),\ (2,3),(4,5),(6,7)\}$, determine if the inverse relation of this set is a function.

Solution Since the second components are all different from one another, the inverse relation is a function.

In fact, the inverse of the set A is given by $A^{-1} = \{(2,1),(3,2),(5,4),(7,6)\}$,
which is clearly a function, since the first components are all different from one another.

An example of a function whose inverse relation is **not** a function is the set $\{(3,4),(5,4),(6,5)\}$**,** because the first and the second ordered pairs have the same second component , 4.

Case 2: Given the tabular form of the function

If any two *y*-values are the same, then the inverse relation is not a function (but only a relation) otherwise, it is a function. This case corresponds to the set form of a function.

Example 4 **Table 3** represents a function whose inverse relation is a function, while **Table 4** represents a function whose inverse relation is **not** a function.

Tables 3

x	y
1	2
2	3
3	4
4	5
5	6
6	7

Table 4

x	y
1	2
2	**3**
3	**3**
4	5
5	7
6	8

} same two *y*-values

Case 3: Given the graphical form of the function

The horizontal line test

If any **horizontal line** drawn to intersect the graph intersects the graph at only one point, then the inverse relation of the function (graph) is a function. However, if a horizontal line meets the graph in more than one point, then the inverse of the graph is not a function but only a relation. See Figs 1, 2 and 3 below/ .

Example 3 Figure **1** is **not** the graph of a one-to-one function, Figures **2** and **3** are the graphs of one-to-one functions.

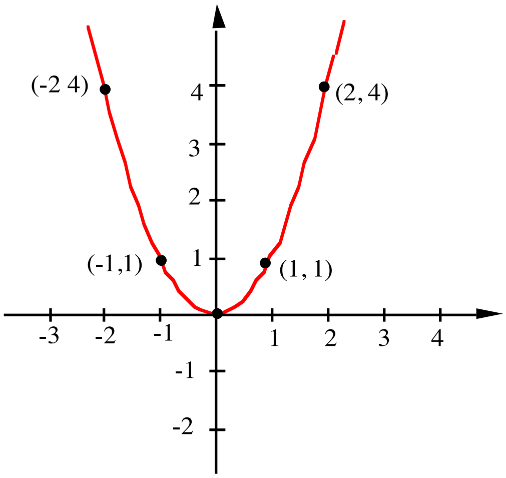


Figure 1: The graph of $y = x^2$,
The inverse relation of this graph is **not** a function

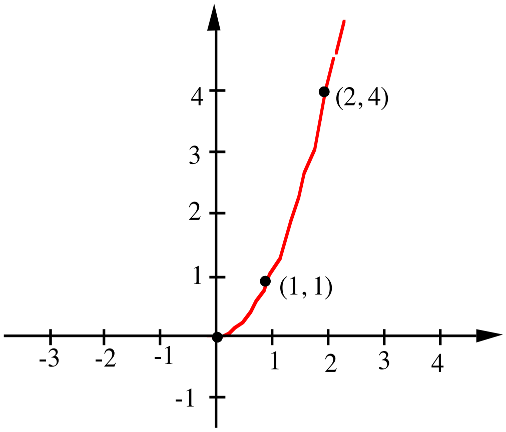


Figure 2: The graph of $y = x^2, \ x \geq 0$.
The inverse relation of this graph is a function
(This is the graph of a one-to-one function)

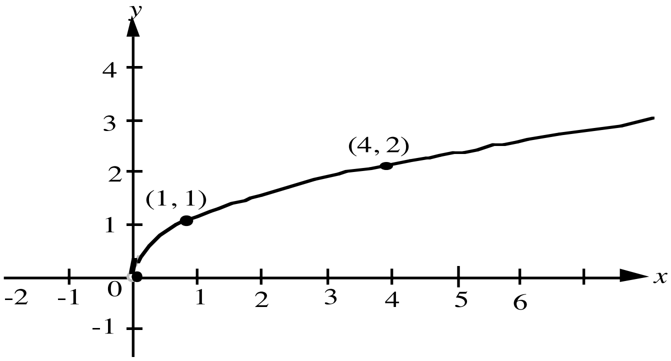


Figure 3: The inverse relation of this graph is a function. (This is the graph of a one-to-one function)

Case 4: Given the equation of the function

By definition, y is a function of x if there is a rule which gives only one corresponding value of y for a given value of x.

Example Determine if the inverse relation of $y = x^2$ is a function.

Method 1: By interchanging the roles of x and y in the given equation, we obtain the inverse relation

$x = y^2$ or $y^2 = x$. Solving $y^2 = x$ for y, we obtain, $y = +\sqrt{x}$ or $y = -\sqrt{x}$

Since for the same x-value (say, 4) we have two different y-values (+2 and -2),. the inverse relation of $y = x^2$ is not a function. However, we could make the inverse relation become a function by restricting the domain of this function (see Figures 1 and 2, above)

For some functions, we will graph the function and use the graphical method.

Method 2: Let $y = f(x)$

Step 1: $f(x_1) = x_1^2$

$f(x_2) = x_2^2$

Equate RHS of $f(x_1)$ to RHS of $f(x_2)$ (That is, let $f(x_1) = f(x_2)$):

$x_1^2 = x_2^2$ (1)

Step 2: Solve for x_1. (You may also solve for x_2)

$x_1^2 = x_2^2$

$x_1 = \pm\sqrt{x_2^2}$

$x_1 = +x_2$ or $-x_2$ (2)

Since from above, $f(x_1) = f(x_2)$, does **not** imply $x_1 = x_2$ (from equation (2)), $f(x)$ is **not** one-to-one and therefore does **not** have an inverse function.

Finding the inverse of a function specified by an equation

Example 3. Find the inverse function of $f(x) = 3x + 2.$ (1)

Solution

Step 1: Let $f(x) = y$. Then $y = 3x + 2.$

Interchange *x* and *y* in the given equation.

Then, we obtain $x = 3y + 2.$ (2)

By tradition, we want to keep the *x*-axis horizontal and express *y* as a function of *x*.

Step 2: Solve equation (2) for *y*.

Then from $x = 3y + 2.$

$$\frac{x-2}{3} = y$$

or $y = \frac{x-2}{3}$

The inverse $f^{-1}(x) = \frac{x-2}{3}$

Alternatively,

Step 1: Solve $y = 3x + 2.$ for *x*.

Then $x = \frac{y-2}{3}$

Step 2: Interchange *x* and *y* (by definition of the inverse)

$$y = \frac{x-2}{3}$$

The inverse $f^{-1}(x) = \frac{x-2}{3}$

We can observe from above that Steps 1 and 2 are interchangeable.

Since the inverse $y = \frac{x-2}{3}$ is also a function, we can say that $y = 3x + 2.$ and $y = \frac{x-2}{3}$ are inverse functions (of each other). We may also add that $f(x) = 3x + 2.$ is invertible.

Example 4 Find the inverse function of $f(x) = x^2 + 6$.

Solution

Step 1: Let $f(x) = y$

Step 2: Solve for x .

$$y = x^2 + 6.$$
$$x = \pm\sqrt{y-6} \qquad (2)$$

Step 3: Interchange x and y in equation (2).

Then $y = \pm\sqrt{x-6}$ (same as $y = +\sqrt{x-6}$ or $y = -\sqrt{x-6}$) (3)

Clearly, equation (3) does not represent a function, since for a given value of x, there are two y-values. We can test this by graphing and using the so called **vertical line tes**t (see page 50). We can say that the given function has an inverse relation specified by $y = \pm\sqrt{x-6}$. However, the given function does not have an inverse function.

In this example, if we were asked to find $f^{-1}(x)$, we would say that there is no $f^{-1}(x)$ (no inverse function) for the given function.

Example 5 Find the inverse function of $f(x) = x^2 + 6 \qquad x \geq 0$

Solution

Step 1: Let $f(x) = y$

Then $y = x^2 + 6 \qquad x \geq 0$

Step 2: Solve for x .

Then $x = \pm\sqrt{y-6}$ (That is, $x = +\sqrt{y-6}$ or $x = -\sqrt{y-6}$) (1)

Since we are only interested in the domain $x \geq 0$, we reject the negative part of equation (1).

Then, we obtain $x = +\sqrt{y-6}$ (2)

Step 3: Interchange (to obtain the inverse) x and y in equation (2)

Then $y = +\sqrt{x-6}$ (3)

Equation (3) is the inverse of $y = x^2 + 6 \qquad x \geq 0$

Therefore, $f^{-1}(x) = \sqrt{x-6} \qquad x \geq 6$

The graph of this inverse (Figure **2**) indicates that the inverse is a function.

Similarly, if the given function were $f(x) = x^2 + 6 \qquad x \leq 0$,

the inverse relation would be $f(x) = -\sqrt{x-6}$ and this also is a function.

ALTERNATIVELY

Step 1: Let $f(x) = y$. Then $y = x^2 + 6 \qquad x \geq 0$

Step 2: Interchange x and y and solve for y.

$$x = y^2 + 6 \qquad y \geq 0$$
$$y = \pm\sqrt{x-6} \qquad y \geq 0$$
$$y = \sqrt{x-6} \quad \text{since} \quad y \geq 0$$

Therefore, $f^{-1}(x) = \sqrt{x-6} \qquad x \geq 6$

We may note from Examples 2 and 3 that, sometimes, by redefining (or restricting) the domain of a given function, a function which does not have an inverse function can be made to have an inverse function. See Figures 1, 2 and 3.

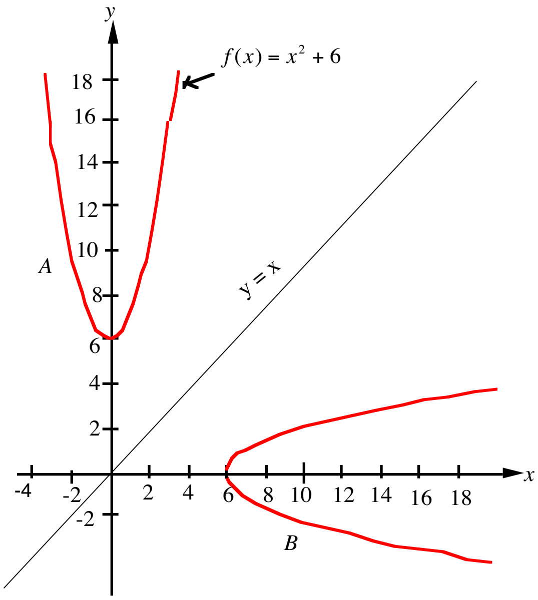


Figure 1:B, the inverse relation of A, is **not** a function .

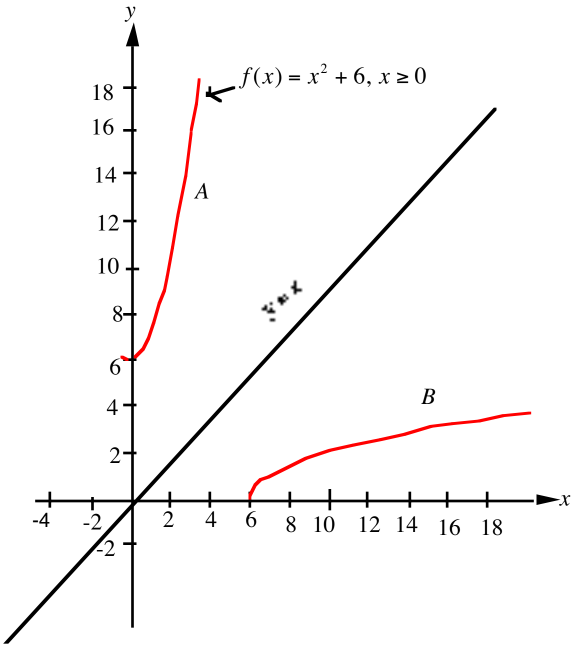


Figure 2: B, the inverse relation of A, is a function (by the vertical line test.)

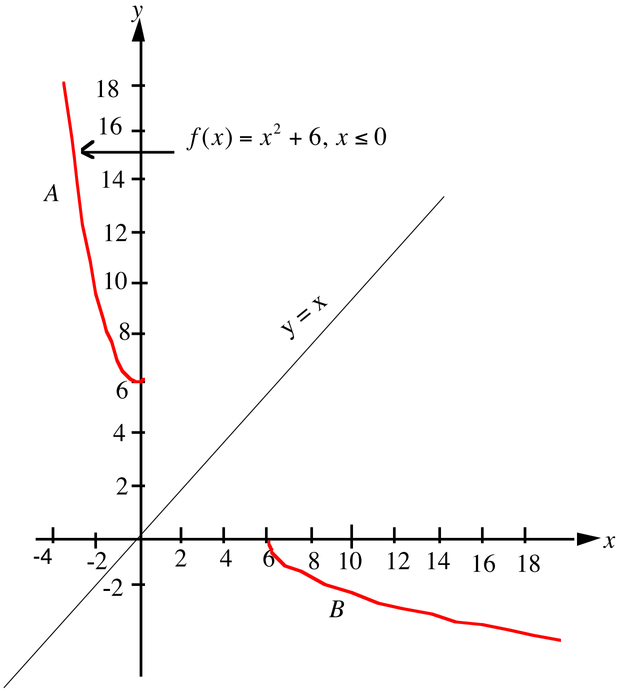


Figure 3: B, the inverse relation of A, is a function (by the vertical line test)

Given the graph of a function, how to sketch the inverse of the graph

Example 5 Given the graph of $y = 3x + 2$, sketch the graph of the inverse of this function.

Geometrically, a function (or relation) and its inverse are symmetric with respect to the line $y = x$. To obtain the inverse graph, we will reflect (see page 223) the given graph about the line $y = x$. The given function is a straight line and so, we will reflect two points about the line $y = x$, and then connect these points by a straight line.

Step 1: Interchange the x- and y-coordinates of any two points on the given line, say, (1, 5) becomes (5, 1); and (-2, -4) becomes(-4, -2).

Step 2: Plot the points (5, 1) and (-4,- 2) on the same coordinate system of axes (as the given graph)

Step 3: Connect the points from Step 2 by a straight line to obtain the inverse graph (Fig.)

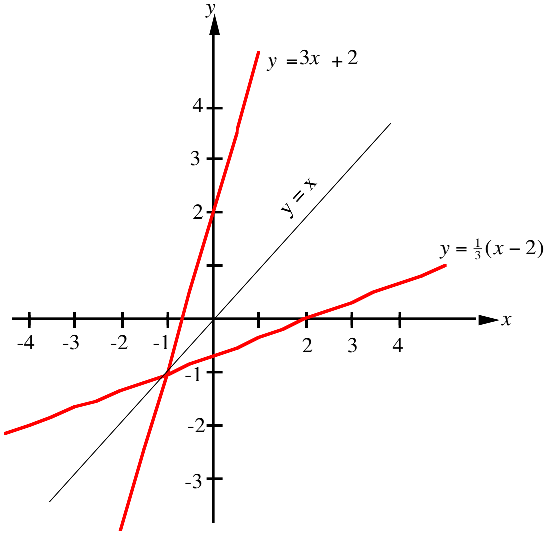


Figure 4: The graphs of $y = 3x + 2$ and $y = \frac{1}{3}(x - 2)$

Note above that, we could also find the equation of the graph, find its inverse equation and then use this equation to sketch the inverse graph.

Determining if two functions are inverses of each other

Case 1: Given the equations of the functions

Example Are the functions $y = 4x + 2$ and $y = \frac{x-2}{4}$ inverses of each other?

Solution

Method 1 We will use the principle of composition of functions which states that:

Two functions f_1 and f_2 are inverses of each other if

1. $f_1[f_2(x)] = x$ and **2**. $f_2[f_1(x)] = x$ (Note that inverse functions reverse the action of each other.)

Let $f_1 = 4x + 2$ and $f_2 = \frac{x-2}{4}$

$$
\begin{aligned}
f_1[f_2(x)] &= 4[\frac{x-2}{4}] + 2 \\
&= \cancel{4}[\frac{x-2}{\cancel{4}}] + 2 \\
&= x - 2 + 2 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
f_2[f_1(x)] &= [\frac{(4x+2)-2}{4}] \\
&= \frac{4x+2-2}{4} \\
&= \frac{4x}{4} \\
&= x
\end{aligned}
$$

Since $f_1[f_2(x)] = f_2[f_1(x)] = x$,

$y = 4x + 2$ and $y = \frac{x-2}{4}$ are inverses of each other.

Method 2 Find the inverse of one of the functions and compare this inverse with the other function.

Step 1: We find the inverse of $y = 4x + 2$ (by interchanging x and y and solving for y)

$x = 4y + 2$

$y = \frac{x-2}{4}$ (solving for y)

Step 2: Clearly, this inverse is identical with the other function.

Therefore, $y = 4x + 2$ and $y = \frac{x-2}{4}$ are inverses of each other.

Case 2: Given the graphs of the functions

Procedure: Fold the page along the line $y = x$ and if the two graphs coincide, then the functions are inverses of each other. See Figures 2 and 3 (page 78, 79,)

Lesson 12 Exercises

A Find the inverse relation in each of the following and state if the inverse relation is a function:

1. $A = \{(2,3),\ (4,5),\ (6,7)\}$ **2**. $B = \{(0,1),\ (2,5),\ (4,10)\}$ **3**. $C = \{(b,a),\ (c,d),\ (a,b)\}$

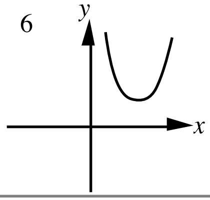
4. $\{(1,3),(2,3),(4,5)\}$

5. Given a table for a function, determine if the inverse relation represents an inverse function.

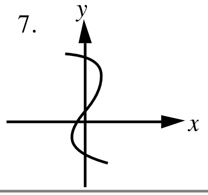
x	2	4	8	
y	5	6	10	1

Determine if the inverse relation in each of the following graphs represents an inverse function:

6



7.



Answers: 1. $\{(3,2),(5,4),(7,6)\}$. It is a function; **2**. $\{(1,0),(5,2),(10,4)\}$ .It is a function.
3. $\{(a,b),(d,c),(b,a)\}$.It is a function; **4**. $\{(3,\ 1),(3,\ 2),(5,\ 4)\}$ .It is not a function.
5. The inverse relation is a function; **6.** The inverse relation is **not** a function;
7. The inverse relation is a function; even though the given relation is not a function.

B Find the inverse relation of the following and indicate which of the inverse relations are functions.

1. $y = -4x + 1$; **2**. $y = x^2 - 3$ **3**. $y - 5 = x^2$; **4**. $y = x^3$; **5**. $x - 2y = 6$

Determine which of the following pairs are inverses of each other.

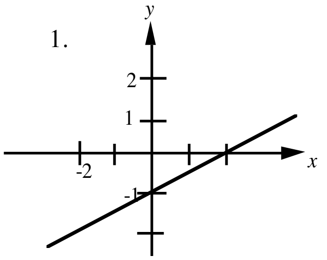
6. $y = \frac{1}{x+2} - 3$ and $y = \frac{1}{x+1} - 4$; **7**. $y = 5x + 2$ and $y = \frac{x-2}{5}$; **8**. $y = x^3$ and $y = x^{\frac{1}{3}}$

Answers: **1**. $y = -\frac{x}{4} + \frac{1}{4}$(a function); **2**. $y = \pm\sqrt{x+3}$ (**not** a function); **3**. $y = \pm\sqrt{x-5}$(**not** a function);

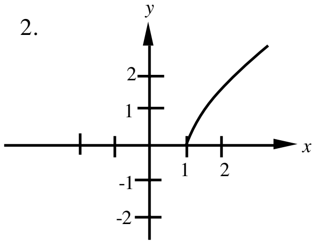
4. $y = x^{\frac{1}{3}}$ (a function); **5**. $y = 2x + 6$ (a function); **6**. No; **7**. Yes; **8**. Yes.

C. In each of the following graphs, sketch its inverse by reflecting the graph in the line $y = x$:

1.

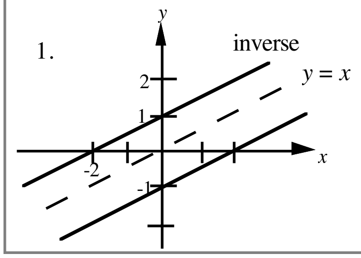


2.

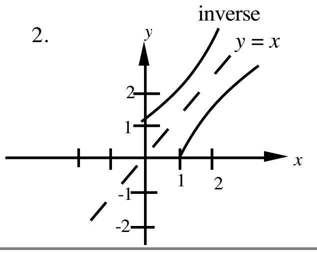


Answers:--

1.



2.



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Introductory Theme for Chapter 7

(Next Chapter)

Straight Line

Theme: Two points.

1. Why two points? Two points, because given or knowing two points, a straight line can be drawn by connecting the two points, using a straight edge and pencil.

2. Why two points? Two points, because given or knowing two points

the slope, m, of the line segment connecting the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ can be found

by applying $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

3a. Why two points? Two points, because if we know the **slope, *m***, and the ***y*-intercept, *b*,** of the line, we can obtain two points and draw the graph of the line

3b Note: y–intercept,b implies the point $(0, b)$, By choosing a point (x, y) on a line, we have two points, and the slope (as well as an equation) of the line connecting the two pints $(0, b)$ and (x, y) is given by

$m = \dfrac{y-b}{x-0}$ <--**slope = slope**

$mx = y - b$ or

$\boxed{y = mx + b}$ <------**slope-intercept form** of the equation of a line

4. Why two points? Two points, because given or knowing two points

an equation of the line segment connecting the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ can be found by

applying $\boxed{y - y_1 = \left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)}$ (from $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$ <-- **is slope = slope**

or $\boxed{y - y_1 = m(x - x_1)}$ <------**point-slope form,** where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

5. Why two points? Two points, because given or knowing the two-intercept points $(a, 0)$, $(0, b)$ an equation of the line segment connecting the two points $P_1(a, 0)$ and $P_2(0, b)$ can be found by

applying $y - y_1 = \left(\dfrac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$ to obtain

$y - 0 = \dfrac{b-0}{0-a}(x - a)$ (or from $\dfrac{y-0}{x-a} = \dfrac{b-0}{0-a}$) <--- **slope = slope**

or $y = \frac{b}{-a}(x - a)$ or $y = \frac{b}{-a}x + (\frac{b}{-a})(-a)$ or $\boxed{y = -\frac{b}{a}x + b}$ also $\frac{x}{a} + \frac{y}{b} = 1$ **(Two intercept form)**

6. Why two points? Two points, because given the graph (picture) of a line , we are given infinitely many points from which we can read the coordinates of any two points on the line and write an equation of a line by applying **3, 4** or **5** above. A picture is worth a thousand words.

The above theme summarizes Chapter 7

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CHAPTER 6

Trigonometrical Functional Value of any Angle

Lesson 18: **Definitions; Functional Value Given the Measure of the Angle**
Lesson 19: **Given a Point on the Terminal Side of the Angle**
Lesson 20: **Trigonometric Functional Values of Quadrantal Angles**

So far, we have defined the trigonometric ratios (functions) in terms of the sides of a right triangle and it did not matter where the vertex and the initial side of the angle were located, provided we obtained the required angular rotation. In this chapter, we will define two additional terms so that we can apply the trigonometric functions much more widely and we will not be restricted to angles between 0° and 90°.

Lesson 18

Basic Definitions; Functional Values given the Measure of the Angle

Basic Definitions

Angle

In trigonometry, we define an **angle** as being formed by rotating a line segment (a ray) about one of its end points, from an initial position to a final or terminal position. We call the end point, the **vertex**. We call the initial position, the initial side and the terminal position, the terminal side. The angle formed is measured by the amount of rotation. We shall refer to the angles formed by these rotations as **directed angle**s. By a directed angle we mean if the rotation (used in forming the angle) is in a **counterclockwise direction**, then we shall say that the measure of the angle formed is **positive** (Figure. 7). However, if the rotation is in a **clockwise direction**, then we say that the measure of the angle formed is **negative** (Figure 8). A curved arrow is used to denote the direction and amount of rotation. If no direction is indicated, then it is to be understood that the angle involved is the smallest positive angle.

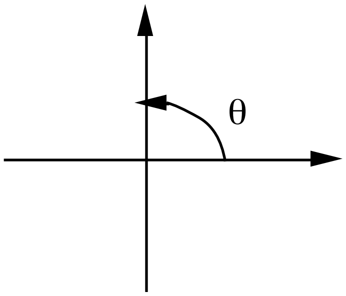


Fig. 7: θ is a positive angle

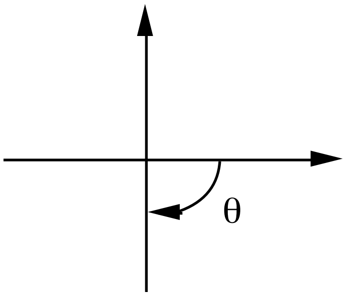


Fig. 8: θ is a negative angle

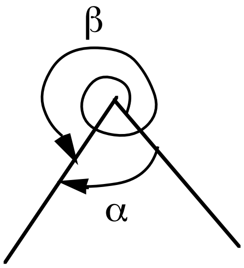
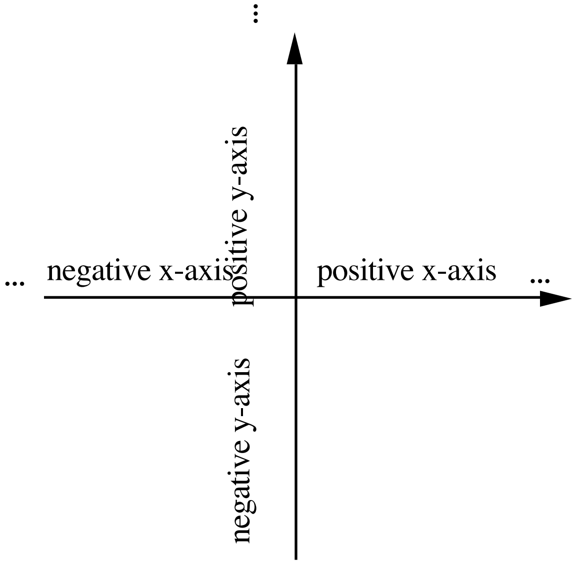


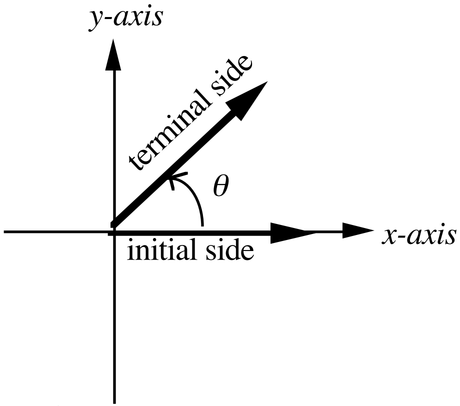
Fig. 9: β is a positive angle
α is a negative angle

A ray or a side can be rotated clockwise or counter-clockwise indefinitely. Consequently, the measure of an angle has an unlimited number of values. However, with some restrictions we can have unique values for the angles.

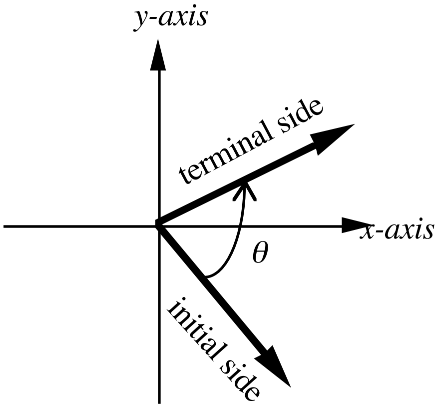
Angle in standard position

Definition: An angle is in standard position in an *x*-*y* rectangular coordinate system if it's vertex is at the origin and its initial side is along positive *x* -axis.

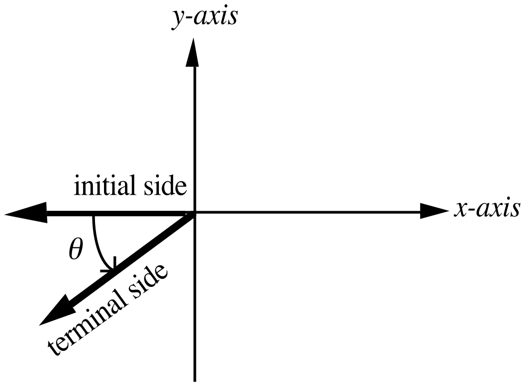




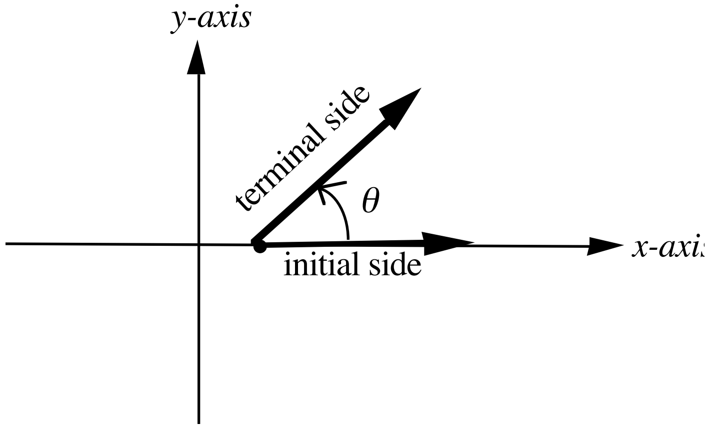
Angle in standard position



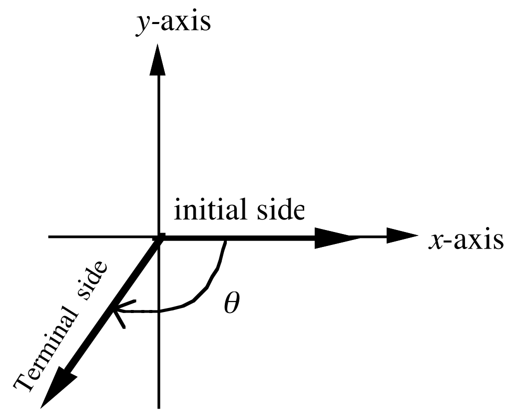
Angle **not** in standard position
(Initial side not along the positive *x*-axis)



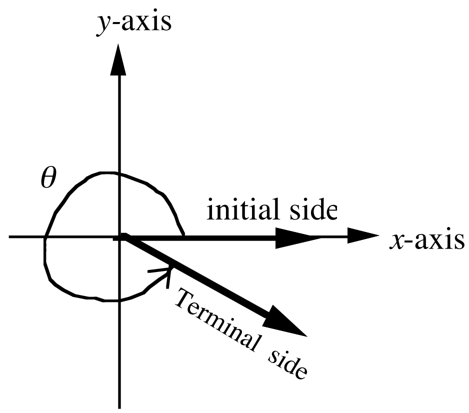
Angle **not** in standard position
(Initial side not along the positive *x*-axis)



Angle **not** in standard position
(Vertex not at the origin)



Angle in standard position



Angle in standard position

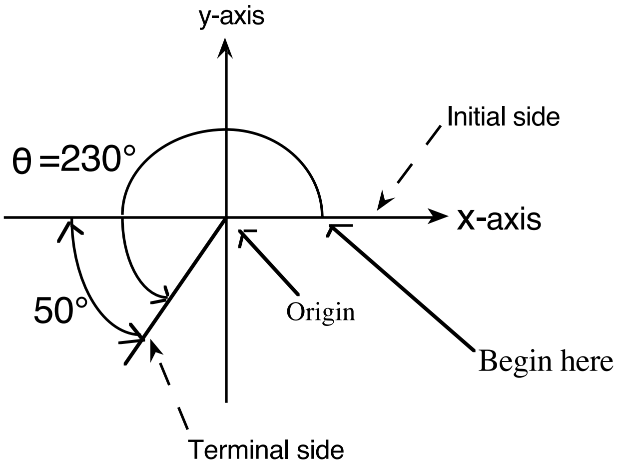
Reference Angle

Definition: If an angle, θ, is in standard position, then the reference angle is the positive acute angle between the terminal side of θ and the x- axis (the horizontal axis).

Example 1 Find the reference angle for 230°.

Solution

Step 1: Draw (or sketch) $\theta = 230^\circ$ in standard position.
To draw the arc, begin from the positive x-axis and draw the arc counterclockwise.



Step 2: Determine the reference angle by difference. The reference angle for 230° is 230 - 180 i.e. 50°

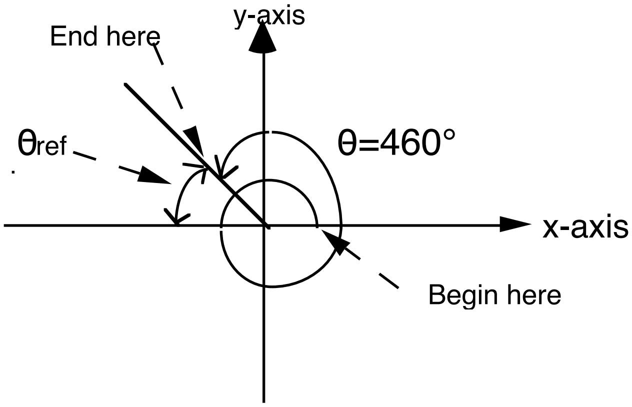
The reference angle for 230° is 50° .

Note: The reference angle is always between the terminal side and the x-axis.

Example 2 Find the reference angle for 460°

Solution

Step 1: Draw (sketch) θ = 460° in standard position.



Step 2: The reference angle is the difference between 180° and 100°.

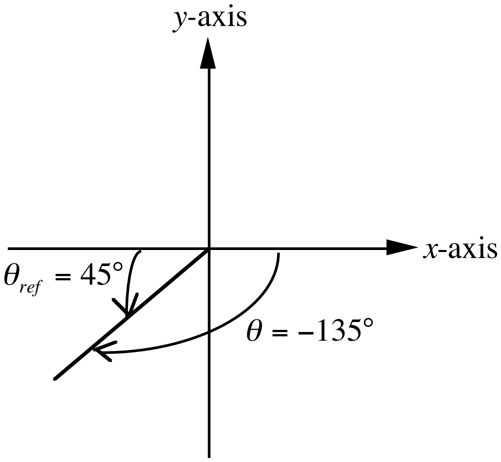
i.e., 180° - 100° = 80°

Note that 360° will bring you back to the initial side, leaving 460 - 360 = 100 to consider.

Example 3 Find the reference angle for $\theta = -135°$

Solution

Step 1: Draw (sketch) θ = -135° (Draw this clockwise, since θ is negative.)



Step 2: The reference angle is $180° - 135° = 45°$, but note that the reference angle is always between the terminal side and the *x*-axis (never between the *y*-axis).

Relationship between θ (standard-position angle) and θ_{ref}

1. If θ is in the **first quadrant**, then $\theta_{ref} = \theta$

2. If θ is in the **second quadrant**, then $\theta_{ref} = 180° - \theta$ or equivalently $\theta = 180° - \theta_{ref}$

3. If θ is in the **third quadrant**, then $\theta_{ref} = \theta - 180°$ or equivalently $\theta = \theta_{ref} + 180°$

4. If θ is in the **fourth quadrant**, then $\theta_{ref} = 360° - \theta = \theta$ or equivalently $\theta = 360° - \theta_{ref}$

Note that θ and θ_{ref} are always in the **same** quadrant.

Trigonometric Functional Value of any Angle (Given the measure of the angle)

Example (a) Find the value of sin 210˚ without using a calculator: use trigonometric tables.
(b) Verify your answer using a calculator.

Solution (a)

Step 1: Draw (or sketch) the given angle in standard position.

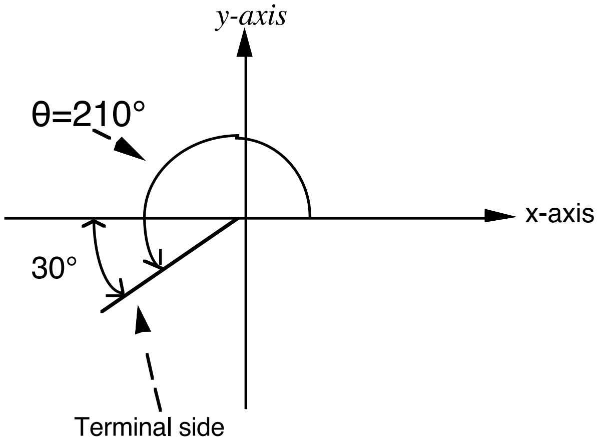


Figure 1

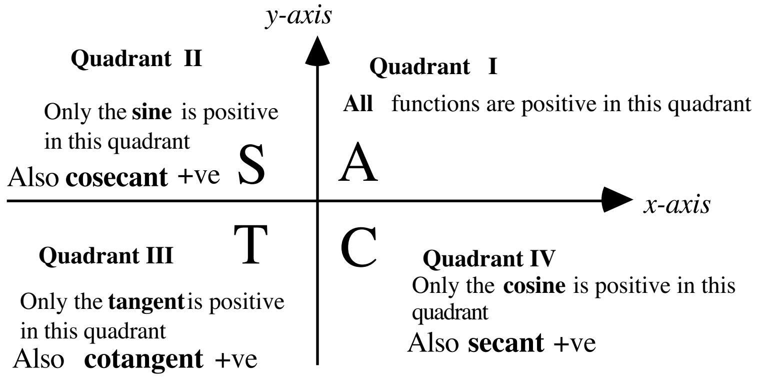
Step 2: Find the reference angle.
The reference angle is 210° - 180° = 30°

Step 3: From tables or from memory, find sin 30˚. This value will be positive, and we must then determine the sign of sin 210˚ according to the quadrant in which the terminal side of this angle lies. (See Figure 2 below) The terminal side of 210° is in the 3rd quadrant where the sine function is negative.

$$\therefore\ \sin 210 = -\sin 30 = -\frac{1}{2}$$

Signs of Trigonometric Functions According to Quadrants

Mnemonic device " **CAST**" for remembering in which quadrant a trigonometric function is positive or negative



Quadrant II

Only the **sine** is positive in this quadrant
Also **cosecant** +ve

Quadrant I

All functions are positive in this quadrant

Quadrant III

Only the **tangent** is positive in this quadrant
Also **cotangent** +ve

Quadrant IV

Only the **cosine** is positive in this quadrant
Also **secant** +ve

Figure 2

In the above device: The **A** in "**CAST**" means **A**ll functions are positive in this quadrant; **S** for only sine is positive in this quadrant; and similarly **T** for tangent; and **C** for cosine.

Note: The order of the above "CAST" is counterclockwise. There are other mnemonic devices such as "**A**ll **S**tudents **T**ake **C**alculus".

(b) With the calculator in degree mode, press 210; then press "the sin" key and read -0.5

Note above: The sign of a trigonometric function and its reciprocal are the same.

Lesson 18 Exercises

A Find the reference angles for the following: **1.** 250° ; **2.** 295°; **3.** 560°; **4.** 75°; **5.** 360°

6 −35°; **7.** −125°; **8.** −265°

Answers. **1.** 70° ; **2.** 65° ; **3.** 20°; **4.** 75° ; **5.** 0°; **6** 35°; **7.** 55°; **8.** 85°.

B In the following problems, first use tables; and then verify your answers using a calculator.

Find the values of the following: **1.** sin 250° ; **2.** cos 250°; **3.** sin 460°; **4.** sin 75°

Answers: **1.** - 0.94 ; **2.** -0.34; **3.** 0.98; **4.** 0.96

C Find the following:

(a) $\sin(-60°)$; (b) $\cos(-60°)$; (c) $\sin(-220°)$; (d) $\cos(-380°)$; (e) $\sin\frac{5\pi}{4}$; (f) $\cos\left(-\frac{\pi}{3}\right)$.

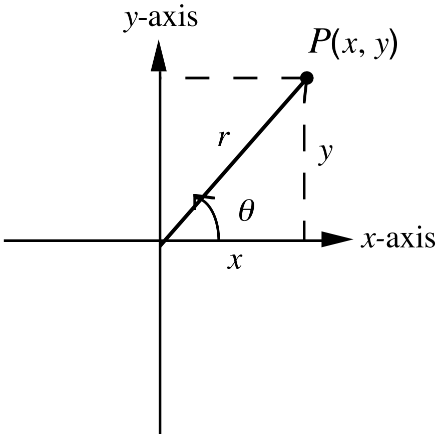
(g) $\sin(-150°)$; (h) $\cos(-250°)$; (i) $\sin(-460°)$; (j) $\sin(-75°)$

Answers: (a) -.86; (b) .5; (c) .64; (d) .94; (e) -.71; (f) .5; (g) -.5; (h) -.34; (i) -.98; (j) -.97

Lesson 19

Trigonometric Functional Value of any Angle

(Given the coordinates of a point on the terminal side of an angle)



Let θ be an angle in standard position in a rectangular coordinate plane. Let $P(x,y)$ be a point on the terminal side of the angle θ and at a positive distance r from the origin (**Figure**).

Applying Pythagorean theorem to the above figure,

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

With reference to θ, r is the hypotenuse, x the adjacent side, y is the opposite side.
We define the six trigonometric functions (see also page 97.) for θ in terms of r, x and y .
(Recall: SOH , CAH, TOA)

$$\sin\theta = \frac{y}{r}\ ; \qquad \csc\theta = \frac{r}{y}.$$

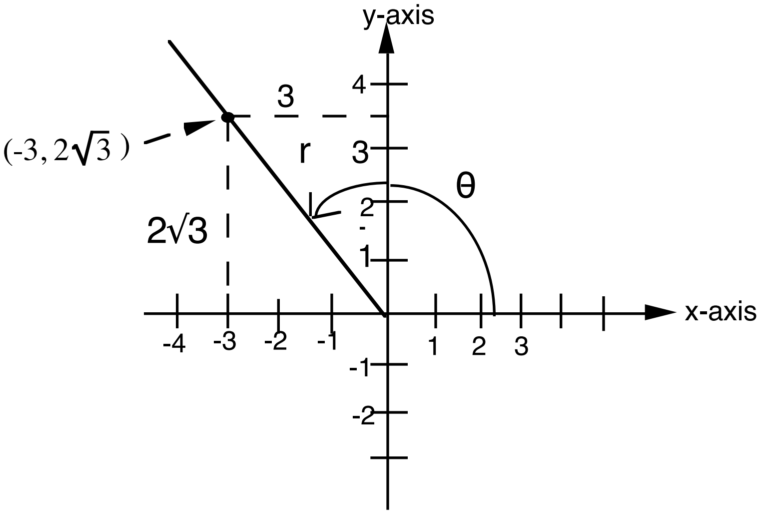
$$\cos\theta = \frac{x}{r}\ ; \qquad \sec\theta = \frac{r}{x}.$$

$$\tan\theta = \frac{y}{x}\ ; \qquad \cot\theta = \frac{x}{y}.$$

Note above that r is always positive while x and y may be positive or negative.

Example Find the six trigonometric values of θ, (if θ is an angle in standard position) given that the terminal side of θ passes through the point $(-3, 2\sqrt{3})$.

Step 1: Draw (or sketch) θ with terminal side passing through the point $(-3, 2\sqrt{3})$
(x-coordinate = -3; y-coordinate = $2\sqrt{3}$)



Step 2: Find r using the Pythagorean Theorem.

$$r^2 = (-3)^2 + (2\sqrt{3})^2 \quad (r^2 = x^2 + y^2)$$
$$r^2 = 9 + 4\sqrt{9}$$
$$r^2 = 9 + 12 \qquad \text{(Note: } 4\sqrt{9} = 4(3) = 12))$$
$$r^2 = 21$$
$$r = \sqrt{21}$$

Step 3: Now, $r = \sqrt{21}$, $x = -3$, $y = 2\sqrt{3}$

(a) $\cos\theta = \dfrac{x}{r}$

$$= \frac{-3}{\sqrt{21}}$$
$$= -\frac{3}{\sqrt{21}} \bullet \frac{\sqrt{21}}{\sqrt{21}} \qquad \text{(Rationalizing the denominator)}$$

$$\therefore \cos\theta = -\frac{3\sqrt{21}}{21}$$

(b) $\sin\theta = \dfrac{y}{r}$ $\qquad (y = 2\sqrt{3},\ r = \sqrt{21})$

$$= \frac{2\sqrt{3}}{\sqrt{21}}$$

$$= \frac{2\sqrt{3}}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}}$$ (Rationalizing the denominator)

$$= \frac{2\sqrt{63}}{21}$$

$$= \frac{2\sqrt{9}\sqrt{7}}{21}$$

$$= \frac{2\cancel{(3)}\sqrt{7}}{\cancel{21}_{7}}$$ $(\sqrt{9} = 3)$

$$\sin\theta = \frac{2\sqrt{7}}{7}$$

(c) $\tan\theta = \frac{y}{x}$ $(y = 2\sqrt{3}, x = -3)$

$$= \frac{2\sqrt{3}}{-3}$$

$$\therefore \tan\theta = -\frac{2\sqrt{3}}{3}$$

(d) $\sec\theta = \frac{1}{\cos\theta}$

$$= \frac{1}{\frac{x}{r}} = \frac{r}{x} = \frac{\sqrt{21}}{-3}$$

$$\therefore \sec\theta = -\frac{\sqrt{21}}{3}$$

(e) $\csc\theta = \frac{1}{\sin\theta}$

$$= \frac{r}{y}$$

$$= \frac{\sqrt{21}}{2\sqrt{3}}$$

$$= \frac{\sqrt{21}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$ (Rationalizing the denominator)

$$= \frac{\sqrt{63}}{2\sqrt{9}}$$

$$= \frac{\sqrt{63}}{2(3)}$$ $(\sqrt{9} = 3)$

$$= \frac{\sqrt{9}\sqrt{7}}{6}$$

$$= \frac{3\sqrt{7}}{6}$$

$$\therefore \csc\theta = \frac{\sqrt{7}}{2}$$

(f) $\cot\theta = \frac{1}{\tan\theta}$

$$= \frac{1}{\frac{y}{x}}$$

$$= \frac{x}{y}$$

$$= \frac{-3}{2\sqrt{3}} \qquad (x = -3, y = 2\sqrt{3})$$

$$= -\frac{3}{2\sqrt{3}} \bullet \frac{\sqrt{3}}{\sqrt{3}} \qquad \text{(Rationalizing the denominator).}$$

$$= -\frac{3\sqrt{3}}{2\sqrt{9}}$$

$$= -\frac{\cancel{3}\sqrt{3}}{2(\cancel{3})} \qquad (\sqrt{9} = 3)$$

$$\therefore \cot\theta = -\frac{\sqrt{3}}{2}$$

Comparatively, note in the last two examples that:
A. On page123 we had to determine the sign of the function according to the quadrant (in which the terminal side of the angle lies) if we obtain the values of the acute angles from tables or from memory.

B.On page125 , we did not have to determine the sign of the function. The signs were obtained solely from the signs of the *x*- and *y*-coordinates; but note also that *r* is always positive.
Note also however that we may use the method in **A** to do the problems referred to in **B**.

Lesson 19 Exercises

A Find the six trigonometric values of θ , if θ is an angle in standard position, given that the terminal side of θ passes through the point (5, 2).

Solutions: $\sin\theta = \frac{2\sqrt{29}}{29}$; $\cos\theta = \frac{5\sqrt{29}}{29}$; $\tan\theta = \frac{2}{5}$; $\csc\theta = \frac{\sqrt{29}}{2}$; $\sec\theta = \frac{\sqrt{29}}{5}$; $\cot\theta = \frac{5}{2}$

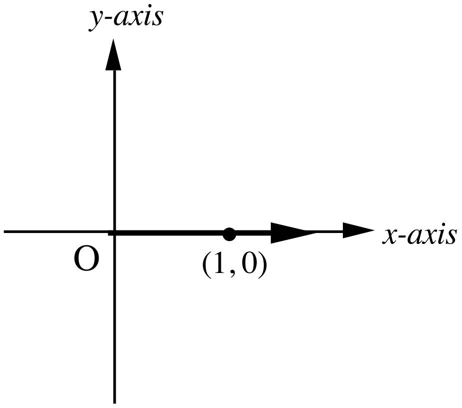
B Redo the Example on page 126 by applying the mnemonic devices SOH, CAH, TOA together with "CAST" referred to on page 123.

Lesson 20

Trigonometric Functional Values of Quadrantal Angles

Quadrantal angles are angles whose terminal sides lie on either the x-axis or the y-axis. These angles are multiples of 90°. Examples of quadrantal angles are 0° , 90°, 180, -180°, 270°, and 360°.

Trigonometric functional values for the quadrantal angle 0°



Consider a point $P(1, 0)$ on the terminal side of an angle $\theta = 0°$ in standard position and its distance from the origin to this point is 1. From the figure, $x = 1$, $y = 0$, and $r = 1$.

$$(a) \quad \sin\theta = \frac{y}{r}$$

$$\sin 0 = \frac{0}{1} \qquad (y = 0,\ r = 1)$$

$$\mathbf{\sin 0° = 0}$$

$$(b) \quad \cos\theta = \frac{x}{r}$$

$$\cos 0° = \frac{1}{1} \qquad (x = 1,\ r = 1)$$

$$\mathbf{\cos 0° = 1}$$

$$(c) \quad \tan\theta = \frac{y}{x}$$

$$\tan 0° = \frac{0}{1} \qquad (y = 0,\ x = 1)$$

$$\mathbf{\tan 0° = 0}$$

The reciprocal relationships are obtained by inverting (a), (b), b and (c) above.

(e) $\csc 0° = \frac{r}{y} = \frac{1}{0}$ **is undefined**.

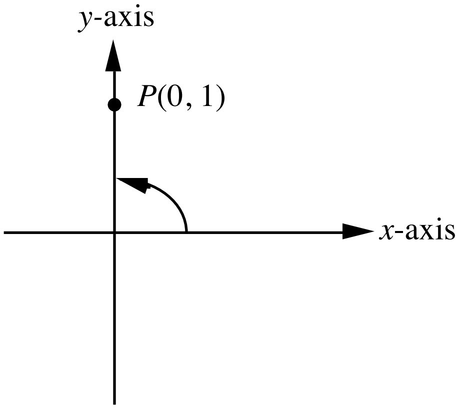
(f) $\sec 0° = \frac{r}{x} = \frac{1}{1}$

$\mathbf{\sec 0° = 1}$

(g) $\cot 0° = \frac{1}{0}$ **is undefined**

Trigonometric functional values for the quadrantal angle 90°

Consider a point $P(0, 1)$ on the terminal side of an angle $\theta = 90°$ in standard position. Then the distance from origin to P is 1. Thus, $r = 1, x = 0$ and $y = 1$.



$$(a) \quad \sin\theta = \frac{y}{r}$$

$$\sin 90° = \frac{1}{1} \qquad (y = 1,\ r = 1)$$

$$\mathbf{\sin 90° = 1}$$

$$(b) \quad \cos\theta = \frac{x}{r}$$

$$\cos 90° = \frac{0}{1} \qquad (x = 0,\ r = 1)$$

$$\mathbf{\cos 90° = 0}$$

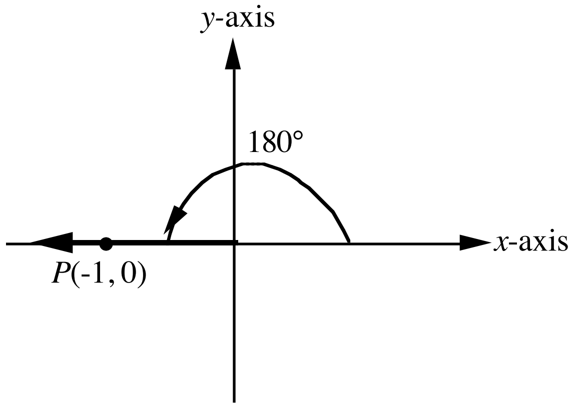
$$(c) \quad \tan\theta = \frac{y}{x}$$

$$\tan 90° = \frac{1}{0} \qquad (y = 1,\ x = 0)$$

tan $90°$ is undefined

To find the functional values for the cosecant, the secant, and the cotangent , invert the right-hand sides of the sine, cosine, and tangent functions, respectively, and deduce the results.

Trigonometric functional values for the quadrantal angle 180°



Consider a point $P(-1, 0)$ on the terminal side of an angle $\theta = 180°$ in standard position. Then the distance from the origin to P is 1. Thus, $r = 1$, $x = -1$ and $y = 0$.

$$(a) \quad \sin 180° = \frac{y}{r}$$

$$\sin 180° = \frac{0}{1} \qquad (y = 0,\ r = 1)$$

$$\mathbf{\sin 180° = 0}$$

$$(b) \quad \cos 180° = \frac{x}{r}$$

$$= \frac{-1}{1} \qquad (x = -1,\ r = 1)$$

$$\mathbf{\cos 180° = -1}$$

$$(c) \quad \tan 180° = \frac{y}{x}$$

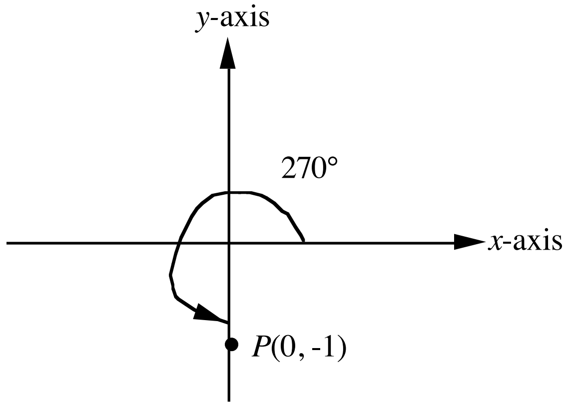
$$= \frac{0}{-1} \qquad (y = 0,\ x = -1)$$

$$\mathbf{\tan 180° = 0}$$

To find the functional values for the cosecant, the secant, and the cotangent , invert the right-hand sides of the sine, cosine, and tangent functions , respectively, and deduce the results.

csc 180° = undefined
sec 180° = −1
cot 180° = undefined

Trigonometric functional values for the quadrantal angle 270°



Consider a point $P(0, -1)$ on the terminal side of an angle $\theta = 270°$ in standard position. Then the distance from origin to P is 1. Thus, $r = 1, x = 0$ and $y = -1$.

$$(a) \quad \sin 270° = \frac{y}{r}$$

$$\sin 270° = \frac{-1}{1} \qquad (y = -1,\ r = 1)$$

$$\mathbf{\sin 270° = -1}$$

$$(b) \quad \cos 270° = \frac{x}{r}$$

$$= \frac{0}{1} \qquad (x = 0,\ r = 1)$$

$$\mathbf{\cos 270°} = 0$$

$$(c) \quad \tan 270° = \frac{y}{x}$$

$$= \frac{-1}{0} \qquad (y = -1,\ x = 0)$$

tan 270° is undefined

To find the functional values for the cosecant, the secant, and the cotangent, invert the right-hand sides of the sine, cosine, and tangent functions, respectively, and deduce the results.

Note In the above derivations, the value of r does not matter.
For example, for $\theta = 0°$, if $r = a, x = a,$ and $y = 0$ we obtain the following:

$$\sin 0° = \frac{y}{r} = \frac{0}{a} = 0$$

$$\cos 0° = \frac{x}{r} = \frac{a}{a} = 1$$

$$\tan 0° = \frac{y}{x} = \frac{0}{a} = 0$$

The trigonometric functional values for the quadrantal angle 360° are the same as those for 0°.

You may also observe that the values for sine, cosine, tangent, cotangent, secant, and cosecant are either -1, 0, 1, or undefined. For sine and cosine, the values are either -1, 0, 1. A mnemonic device for remembering these values is to locate the appropriate axis and choose a convenient " r " (say, $r = 1$) and then apply the fundamental definitions.

Functional Values for Coterminal Angles

Coterminal angles are angles which have the same terminal sides when both angles are placed in standard position on the same coordinate system of axes. Since the location of the terminal side of an angle completely determines the trigonometric functional value, **coterminal angles have the same functional values**.

Examples

(*a*) 270° and $-90°$ are coterminal, and therefore have the same functional value.

(*b*) $-270°$ and 90° are coterminal, and therefore have the same functional value.

(*c*) 180° and $-180°$ are coterminal, and therefore have the same functional value.

(*d*) 30°and 390° are coterminal, and therefore have the same functional value.

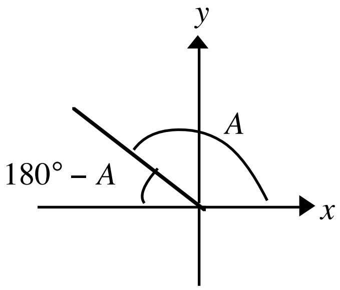
(*e*) 40°and 760° are coterminal, and therefore have the same functional value.

Trigonometric functional values for negative quadrantal angles

Since coterminal angles have the same functional values, the functional values for 270°and $-90°$ are the same, since these two angles are coterminal.

Therefore for $-90°$ see values for 270°. Similarly, for -180° see values for 180° and for $-270°$ see values for 90°.

Functional Values for sin (180° – A), cos (180° – A), and tan (180° – A)



$\sin A = +\sin(180° - A)$ (the sine is positive in the second quadrant)

$\therefore \sin(180° - A) = \sin A$

$\cos A = -\cos(180° - A)$ (the cosine is negative in the second quadrant)

$\therefore \cos(180° - A) = -\cos A$

$\tan A = -\tan(180° - A)$ (the tangent is negative in the second quadrant)

$\therefore \tan(180° - A) = -\tan A$.

Lesson 20 Exercises

Derive the trig functional values for the following:
$0°$, $-180°$, $-270°$

Answers: $\sin 0° = 0$, $\cos 0° = 1$; $\tan 0° = 0$; $\csc 0°$ is undefined; $\sec 0° = 1$; $\cot 0°$ is undefined
$\sin(-180°) = 0$; $\cos(-180°) = -1$; $\tan(-180°) = 0$; $\sec(-180°) = -1$; $\csc(-180°)$ is undefined.; $\cot(-180°)$ is undefined.
$\sin(-270°) = 1$; $\cos(-270°) = 0$; $\tan(-270°)$ is undefined; $\csc(-270°) = 1$; $\sec(-270°)$ is undefined; $\cot(-270°) = 0$.

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CHAPTER 10

Indefinite Integrals and Antiderivatives

Lesson 28

Introduction to Integration

Inverse of Differentiation

In the past, after covering an operation on numbers, we reversed the steps of that operation to obtain a new operation. After the addition operation, we obtained the subtraction operation; after multiplication, we obtained division; after raising numbers to powers, we extracted the roots of numbers; after finding the logarithms of numbers, we found their antilogarithms. Similarly, after covering trigonometric functions, we covered inverse trigonometric functions. Each of the above operations and its reversed operation are inverses of each other. We also used the inverse operation to check the results of a particular operation, since each operation reverses the action of its inverse, For example, we used addition to check the correctness of a subtraction problem, and multiplication to check a division problem. Each operation and its inverse have practical applications.

So also, it would be quite natural that, in calculus, after learning the differentiation of functions, we reverse the steps to obtain the inverse operation which also quite naturally, we call **antidifferentiation** or simply **integration**.

Let us revisit some previous examples covered under differentiation.

Example 1. If $f(x) = x^4 + x^2 + 12$,

then $f'(x) = 4x^3 + 2x + 0$

$= 4x^3 + 2x$

Example 2 If $f(x) = x^4 + x^2 + 5$, then

$f'(x) = 4x^3 + 2x + 0$

$= 4x^3 + 2x$

Observe above that even though the two given functions are different and differ in the constant terms, the two functions have the same derivative, because the derivative of the constant term in each case is zero.

Let us redo Example 1 and then reverse the steps.

Differentiation

If $f(x) = x^4 + x^2 + 12$

$f'(x) = 4x^3 + 2x$

(Exponent **multiplies** the base and the exponent **decreases** by 1)

Integration (Reverse the steps of differentiation)

$\int(4x^3 + 2x)dx$

$= \frac{4x^{3+1}}{3+1} + \frac{2x^{1+1}}{1+1} + C$ (Exponent increases by 1,

$= \frac{4x^4}{4} + \frac{2x^2}{2} + C$ (and new exponent divides.)

$= x^4 + x^2 + C$

Observe that in just reversing the steps in Examples **1** and 2 to obtain the original function, we are unable to obtain the constant terms 12 and 5; and to compensate, we introduce the letter *C,* which is called the constant of integration. To determine the value of the constant of integration we need more information.

In reversing the steps and applying them to a derivative, we are unable to obtain the exact original function, but rather a **family** of functions which differ from one another by the constant term, the integration constant.
We conclude that by merely reversing the steps of the differentiation process, the result (the integral) we obtain is **not** unique. (More examples: $4x^2 + 4$, $4x^2 + 6$, and $4x^2 - 9$ are antiderivatives of $8x$, since the derivative of each of them is $8x$; the only difference between these integrals is the constant term.
We call the "reversed " function the **indefinite integral** of the given function. The indefinite integral is also called the **antiderivative** or the primitive of the given function.
Therefore, given a function $f(x)$, to find the indefinite integral (the antiderivative, or primitive) of $f(x)$ means we are to find another function $g(x) + C$ such that the derivative of $g(x) + C = f(x)$.

Definition: A function $g(x)$ is an antiderivative of a function $f(x)$ if the derivative of $g(x)$ is $f(x)$. (We use $g(x)$ instead of $F(x)$ for pronunciation simplicity)
We symbolize that we are finding the indefinite integral by:

$$\int f(x)\,dx = g(x) + C \qquad \left(\text{where } \frac{d}{dx}\big[g(x)\big] = f(x)\right)$$

Read " the indefinite integral of $f(x)$ is $g(x) + C$ "
The left side of this equation says " find the indefinite integral of $f(x)$ ", and this is followed by " is $g(x) + C$. The symbol $\int$ is called an integral sign, $f(x)$ is called the integrand (the expression to be integrated) and the symbol dx indicates that we are to integrate with respect to x. The functions $g(x) + C$ are the antiderivatives of $f(x)$. When the constant of integration, $C = 0$

$$\int f(x)\,dx = g(x)$$

Note also that $\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$ (differentiation "undoes" integration).
That is, if we integrate a function and differentiate the result, we recover the original function, (This is similar to $(\sqrt[3]{8})^3 = 8$, since finding the cube root followed by cubing recovers the 8). Therefore, after obtaining an antiderivative of a function, we can check this result by differentiating the antiderivative to see if we obtain the original function, the integrand. If we obtain the original integrand, the antiderivative is correct, otherwise it is incorrect.

There is another type of integral called the definite integral, symbolized $\int_a^b f(x)dx$, where a and b are called the limits of integration. If we can find the indefinite integral, finding the definite integral is straightforward and a matter of numerical evaluation. We will not cover the definite integral in this chapter, but we will do so in a different chapter, later.
A simple example on finding the indefinite integral is presented below, followed by examples involving various functions.

Example Find $\int x^3 dx$ **Solution** $\int x^3 dx = \frac{x^4}{4} + C$.

Lesson 29

Integration of Polynomial Functions

Case 1: Integration of a Monomial Function

We symbolize the indefinite integral (also called the primitive, or the antiderivatives) by $\int f(x)dx$, where $f(x)$ is called the integrand, and dx identifies the independent variable. We define the **indefinite integral** of a function $f(x)$ as a function $g(x)+C$ such that the derivative of $g(x)$ equals $f(x)$, and where *C* is an arbitrary constant.

Power Rule for Integration

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$, where *C is* the integration constant, and $n \neq -1$).

Example 1 Find $\int x^2 dx$

We are to find the family of functions, $g(x)+C$, such that the derivative of each function is x^2. It also means we are to find an antiderivative or a primitive or an indefinite integral of x^2.

Solution $\int x^2 dx = \frac{x^{2+1}}{2+1} + C$ (applying the power rule, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$)

$$= \frac{x^3}{3} + C$$

Let us check the integration by differentiation

$$\frac{d}{dx}\left(\frac{x^3}{3}\right) + \frac{d}{dx}(C)$$
$$= \frac{1}{3}\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}(C)$$
$$= \frac{1}{3}(3x^2) + 0$$
$$= x^2$$

We obtain the original integrand.

Therefore, $\int x^2 dx = \frac{x^3}{3} + C$.

Example 2 Find $\int x^3 dx$

Solution

$\int x^3 dx = \frac{x^{3+1}}{3+1} + C$ (applying the power rule, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$)

$$= \frac{x^4}{4} + C.$$

Case 2: Integration of an integrand containing a constant factor

$$\int af(x)dx = a\int f(x)dx$$

Example Find $\int 5x^3dx$

$$\int 5x^3dx = 5\int x^3dx$$ (Factor out the constant factor, and integrate.)

$$= 5\left(\frac{x^{3+1}}{3+1}\right) + C$$

$$= \frac{5x^4}{4} + C.$$

Note:

$$\int dx = \int 1dx$$

$$= x + C$$

$\int 4dx = 4x + C$ (You may view $\int 4dx$ as $\int 4x^0dx = 4x^{0+1} + C = 4x + C$)

Case 3: Integration of a Polynomial Function

Approach: Integrate each term of the polynomial.

$$\int [f(x) + \int g(x)]dx = \int f(x)dx + \int g(x)dx$$

Example Find $\int (4x^3 + 2x^2 - 8x + 7)\,dx$

Solution

$$\int (4x^3 + 2x^2 - 8x + 7)\,dx$$

$$= \int 4x^3dx + \int 2x^2dx - \int 8x\,dx + \int 7dx$$

$$= \frac{4x^{3+1}}{3+1} + \frac{2x^{2+1}}{2+1} - \frac{8x^{1+1}}{1+1} + 7x + C$$

$$= \frac{4x^4}{4} + \frac{2x^3}{3} - \frac{8x^2}{2} + 7x + C$$

$$= x^4 + \frac{2x^3}{3} - 4x^2 + 7x + C.$$

Lesson 29A Exercises

1, Complete the power rule for polynomials $\int x^n dx = ?$

2 Find $\int x^2 dx$

3. Find $\int x^3 dx$

4. Find $\int 5x^3 dx$

5. Find $\int (4x^3 + 2x^2 - 8x + 7)\, dx$

6. $\int 4\, dx$

Answers: **1.** $\frac{x^{n+1}}{n+1} + C$, **2.** $\frac{x^3}{3} + C$; **3.** $\frac{x^4}{4} + C$; **4.** $\frac{5x^4}{4} + C$;

5. $x^4 + \frac{2x^3}{3} - 4x^2 + 7x + C$; **6.** $4x + C$

Lesson 29B Exercises

1, Complete the power rule for polynomials $\int x^n dx = ?$

2 . Find $\int x^3 dx$

3. Find $\int x^4 dx$

4. Find $\int 6x^2 dx$

5. Find $\int (2x^4 + 2x^3 - 8x^2 + 2x - 3)\, dx$

6. $\int 5\, dx$

Answers: **1.** $\frac{x^{n+1}}{n+1} + C$, **2.** $\frac{x^4}{4} + C$; **3.** $\frac{x^5}{5} + C$; **4.** $2x^3 + C$;

5. $\frac{2x^5}{5} + \frac{x^4}{2} - \frac{8x^3}{3} + x^2 - 3x + C$; **6.** $5x + C$.

Lesson 30

General Substitution Techniques of Integration

An analytical and a guided approach to studying substitution (or change of variable) techniques in integration of functions is introduced in this lesson.
A good number of functions can be integrated using simple substitution. Each integrand may be a single composite function, the product of two functions or the quotient of two functions. The basic functions involved include polynomial, rational, radical, exponential, logarithmic, and trigonometric functions. We will cover a guide which will help us to determine quickly if a simple substitution technique will work and also which function to substitute for. Such determination will help reduce or avoid the number of trial-and-error attempts in these techniques.

To facilitate communication, we will classify the substitution (change of a variable) methods as **simple *u*-substitution** (or *t*-substitution**;** any other variable can be used) **or multiple** substitution. We will further divide simple *u*-substitution, into a **one-step** *u*-substitution and a **two-step** *u*-substitution. In multiple substitution, we apply simple substitution more than once using different variables. Another substitution technique called **trigonometric substitution** will **not** be covered in this lesson, but will be covered in a different chapter, later. In multiple substitution, we will apply simple *u*-substitution followed by trigonometric substitution.

To determine if simple u-substitution will work, the following **guidelines** will be helpful.
For communication purposes, we classify simple *u*-substitution into two main types, namely Type 1 (one-step *u*-substitution) and Type 2 (two-step *u*-substitution)

Guidelines for Type 1 (one-step) *u*-substitution

Case 1: Condition for a single composite function
The degree of the "inside" function must be 1 (a linear function).
For example, $\int(2x-1)^4dx$ satisfies this condition but $\int(2x^2-1)^4dx$ does **not** satisfy this condition.
We can therefore apply simple u-substitution to find $\int(2x-1)^4dx$.

Case 2: Conditions for other functions for Type 1 substitution
These two conditions must be satisfied simultaneously.

Condition 1. A given integrand is the product or quotient of functions involving two functions u and u' (read "u-prime") such that the derivative of u equals u' or the derivative of u differs from u' by only a constant factor. For functions whose basic components are polynomials, the degree of u' is 1 less than the degree of u.

Condition 2. The function u' (the derivative) **must** always be in the numerator, but u may be in the numerator or in the denominator (i.e., u' **cannot** be in the denominator).

Note: The integrands in **A**, **B** and **E**, below, satisfy the above two conditions, but the integrand in **C** does **not** satisfy the condition guidelines because the derivative part, $4x^2, (u')$ is in the denominator. That is, simple u-substitution method will work for **A** and **B** and **E** but not for **C.** (See below**.**)

A: $\int \frac{4x^2}{\sqrt{x^3}+1}\,dx$: **B:** $\int 4x^2\sqrt{x^3+1}\,dx$: **C:** $\int \frac{\sqrt{x^3+1}}{4x^2}\,dx$: **E:** $\int x^2(4x^3+2)^5\,dx$

Note also that for functions involving simple **logarithmic functions** such as $\ln x$, if the $\ln x$ part is chosen as u, the derivative $\frac{1}{x}$ will also be in the numerator. However, on simplifying, the "x" will end up in the denominator. Therefore, we can say that the "x" in $\frac{1}{x}$ must be in the denominator. The substitution method will work for **D** and **E** but not for **F.** (See below**.**)

D: $\int \frac{\ln x}{x}\,dx$; **E:** $\int \frac{1}{x\ln x}\,dx$; **F:** $\int \frac{x}{\ln x}\,dx$: (Note $\frac{\frac{1}{x}\ln x}{1} = \frac{\ln x}{x}$ but $\frac{\ln x}{\frac{1}{x}} = x\ln x$)

Note also: **D** $= \int \frac{\ln x}{x}\,dx = \int \frac{1}{x} \bullet \ln x\,dx$; **E** $= \int \frac{1}{x\ln x}\,dx = \int \frac{\frac{1}{x}}{\ln x}\,dx$

Extra: Simple u-substitution method will **not** work for $\int x\ln x\,dx$. Why?

Guidelines for Type 2 (Two-Step) *u*-substitution

Let $f(x) = \frac{g(x)}{h(x)}$. then a two-step u-substitution will work if the following conditions are satisfied simultaneously..

Condition 1: $h(x)$ is composite with the "inside function" being a linear function (degree 1). Usually $h(x)$ is a power of a linear function.

Condition 2: $g(x)$ is a polynomial function

In the **two-step** u-substitution, the first step expresses the denominator and dx in terms of u.

The second step expresses the rest of the integral in terms of u only.

Examples for a two-step u-substitution are $\int \frac{x+6}{(x+4)^3}dx$, $\int \frac{x^2+5}{(x+4)^3}dx$ and $\int \frac{x^2+4}{\sqrt{x+4}}dx$. (See Case 3b of Lesson 31; and Example 7 of Lesson 32)

To excuse the author's "dogmatism" in the Type-2 substitution guidelines, it may be added that perhaps, there may be other functions which may fall under Type-2 substitution.

It must also be noted above that the author has attempted to help guide the student with respect to some popular integrals and that in some cases, the student may have to try and err to determine if simple u-substitution will work. The author's philosophy is that some helpful guidelines are better than no guidelines at all.
After millions of years, experience is still the best teacher.

More Representative Examples for Simple U-substitution

Polynomial Functions

1. $\int x^2(4x^3+2)^5 dx$

Rational Functions

2. $\int \frac{5x}{\left(x^2+1\right)^3} dx$

3. $\int \frac{x}{x^2+2} dx$

4. $\int \frac{x}{\left(x^2+2\right)^3} dx$

5. $\int \frac{x^2}{\left(4x^3+2\right)^5} dx$

Integrands Involving Radicals

6. $\int \frac{4x^2}{\sqrt{x^3+1}} dx$;

7. $\int 4x^2\sqrt{x^3+1}\, dx$;

8. $\int \frac{x+1}{\sqrt{x^2+2x+2}} dx$

9. $\int \frac{\sqrt{x+1}}{1-x} dx$

10. $\int \sqrt[3]{2x-3} dx$

.

11. $\int \frac{\sqrt{x}}{\sqrt[3]{x}+1} dx$

12. $\int \frac{3x}{\sqrt{1-x}}\, dx$

13. $\int x\sqrt{x+3}\, dx$

Exponential Functions

14. $\int \sqrt{1+e^x}\, dx$;

15. $\int \frac{e^x}{1-e^x}\, dx$;

16. $\int \frac{e^{2x}}{1+e^{2x}}\, dx$

Logarithmic Functions

17. $\int \frac{\ln x}{x}\, dx$

18. $\int \frac{1}{x \ln x}\, dx$

19. $\int \frac{\ln x^2}{x} dx$

Trigonometric Functions

20. $\int \sin x \cos x\, dx$

21. $\int \sin^3 x \cos x\, dx$

22. $\int \cos^4 x \sin x\, dx$

23. $\int \cos^2 x \sin x\, dx$4..

24. $\int \tan x \sec^2 x\, dx$.

25. $\int \cot x \csc^2 x\, dx$

26. $\int \sec 3x \tan 3x\, dx$

27. $\int \frac{\sin x}{\cos x} dx$ $(\tan x)$

28. $\int \frac{\cos x}{\sin x} dx$ $(\cot x)$

29. $\int \frac{\sin x}{\cos^2 x}\, dx$

30. $\int \frac{x+6}{(x+4)^3} dx$.

31. $\int \frac{x^2+5}{(x+4)^3} dx$

32. $\int \frac{x^2+4}{\sqrt{x+4}} dx$

Multiple Substitution

33. $\int \sqrt{\frac{1-x}{1+x}}\, dx$

(u-sub., plus trig. sub.)

The rest of this lesson will be devoted to applying simple u-substitution to polynomial integrands. Thereafter, in subsequent lessons, we will cover more examples involving various functions, some of which are listed in the above table. **Students** should master **simple *u*-substitution** techniques early in the study of integral calculus to the extent that they can readily spot its applicability in unexpected places.

Solved (or Worked) Examples on Simple U-Substitution

We will briefly cover two examples for polynomials. The first example (Method 2) is to show the "power" of simple u-substitution, although this method is an overkill for this example. The second example shows how we can avoid tedious expansion by using simple u-substitution. After these two examples on polynomials, we will continue with the integration of the other functions, and when the need arises, and conditions have been satisfied, we will apply simple u-substitution.

On Polynomials

Example 1: Find $\int x^2(4x^3+2)\,dx$

Method 1 (Usual method)

In $\int x^2(4x^3+2)\,dx$, we normally multiply to obtain $\boxed{\int(4x^5+2x^2)\,dx}$ and then integrate term-by term:

$$\int(4x^5+2x^2)\,dx = 4\frac{x^6}{6}+2\frac{x^3}{3}+c=\frac{2}{3}x^6+\frac{2}{3}x^3+c$$

We differentiate to check: $\frac{d}{dx}\left(\frac{2}{3}x^6+\frac{2}{3}x^3+c\right)=\frac{2}{3}(6)x^5+\frac{2}{3}(3)x^2\boxed{=4x^5+2x^2}$

$$\therefore\ \int(4x^5+2x^2)\,dx=\frac{2}{3}x^6+\frac{2}{3}x^3+C$$

Method 2 Simple u-substitution.

Step 1: Let $u=4x^3+2$. Then $\frac{du}{dx}=12x^2$ and from which $dx=\frac{du}{12x^2}$

Step 2: Substitute u for $4x^3+2$; $\frac{du}{12x^2}$ for dx in $\int x^2(4x^3+2)\,dx$.

Then $\int x^2u\frac{du}{12x^2}=\frac{1}{12}\int u\,du=\frac{1}{12}\bullet\frac{u^2}{2}+c=\frac{u^2}{24}+C$

Step 3: Substitute $4x^3+2$ for u. Then $\int x^2(4x^3+2)dx=\frac{(4x^3+2)^2}{24}+C$

$$=\frac{16x^6+16x^3+4}{24}+C\ =\frac{2}{3}x^6+\frac{2}{3}x^3+\frac{1}{6}+c_1\ =\frac{2}{3}x^6+\frac{2}{3}x^3+C$$

Example 2. Find $\int x^2(4x^3+2)^5dx$

Solution Using simple **u-substitution**

In $\int x^2(4x^3+2)^5dx$, we can expand $x^2(4x^3+2)^5$ and then integrate, However we can avoid the tedious expansion and then integrate by using simple u-substitution

Step 1: Let $u=4x^3+2$. Then $\frac{du}{dx}=12x^2$ and from which $dx=\frac{du}{12x^2}$

Step 2: Substitute u for $4x^3+2$; $\frac{du}{12x^2}$ for dx in $\int x^2(4x^3+2)^5dx$.

$$\text{Then } \int x^2u^5\frac{du}{12x^2}=\frac{1}{12}\int u^5\,du$$
$$=\frac{1}{12}\bullet\frac{u^6}{6}+c$$
$$=\frac{u^6}{72}+C.$$

Step 3: Substitute $4x^3+2$ for u.

Then $\boxed{\int x^2(4x^3+2)^5 = \frac{(4x^3+2)^6}{72} + C}$

Let us differentiate to check: $\frac{d}{dx}\left[\frac{(4x^3+2)^6}{72} + c\right]$

$$= \frac{1}{72}\left[6(4x^3+2)^5(12x^2)\right] + 0$$

$$= \frac{1}{72}\left[72x^2(4x^3+2)^5\right]$$

$= x^2(4x^3+2)^5$ <this checks with the original integrand

Now, try by first expanding $x^2(4x^3+2)^5$ and then integrating. (Nice punishment)

Lesson 30 Exercises A

1. Determine by inspection which of the following we can use simple *u*-substitution to integrate,

a. $\int \frac{4x^2}{\sqrt{x^3+1}}\,dx$; **b.** $\int 4x^2\sqrt{x^3+1}\,dx$; **c.** $\int \frac{\sqrt{x^3+1}}{4x^2}\,dx$; **d.** $\int x^2(4x^3+2)^5\,dx$

e. $\int (2x-1)^4\,dx$

2. Find $\int x^2(4x^3+2)\,dx$; **3.** . Find $\int x^2(4x^3+2)^5\,dx$

4. Find $\int \frac{x+1}{\sqrt{x^2+2x+2}}\,dx$

Answers: **1.** a, b, d, and e. **2.** $\frac{2}{3}x^6 + \frac{2}{3}x^3 + C$; **3.** $\frac{(4x^3+2)^6}{72} + C$;

4. $\sqrt{x^2+2x+2} + C$

Lesson 30 Exercises B

1. Determine by inspection which of the following we can use simple *u*-substitution to integrate,

a. $\int \frac{4x^3}{\sqrt{x^2+1}}\,dx$; **b.** $\int 4x^3\sqrt{x^4+1}\,dx$; **c.** $\int \frac{\sqrt{x^2+1}}{4x^3}\,dx$; **d.** $\int x^3(4x^4+2)^5\,dx$

e. $\int (2x-1)^5\,dx$; **f.** $\int x^4(4x^3+2)^5\,dx$; **g.** $\int \frac{4x^2}{\sqrt{x^3+1}}\,dx$

2. Find $\int x^4(4x^5+2)\,dx$; **3.** . Find $\int x^3(4x^4+2)^9\,dx$

4. Find $\int \frac{x+2}{\sqrt{x^2+4x+2}}\,dx$

Answers: **1.** b, d. e, g. **2.** $\frac{(4x^5+2)^2}{40} + C$ or $\frac{2}{5}x^{10} + \frac{2}{5}x^5 + C$;

3. $\frac{(4x^4+2)^{10}}{160} + C$; **4.** $\sqrt{x^2+4x+2} + C$

Lesson 31

Integration of Rational Functions I

(Integrand is a Rational Expression)

There are a number of methods for integrating rational expressions. Each method depends on the type of rational expression. Below, we briefly cover types of rational expressions.

Definitions

A **rational expression** is an expression which is the ratio of two polynomials, A rational expression may be **proper** or **improper.** In addition, a rational expression may also be **irreducible** or **reducible.**

In a **proper rational** expression, the degree of the numerator polynomial is less than the degree of the denominator polynomial.

Example $\frac{x+3}{x^2-4}$ (Degree of numerator is 1; degree of denominator is 2.)

In an **improper rational** expression, the degree of the numerator polynomial is greater than or equal to the degree of the denominator polynomial.

Examples 1. $\frac{x^2-9}{x-2}$ (Degree of numerator is 2; degree of denominator is 1.)

2. $\frac{x}{x+2}$ Degree of numerator is 1; Degree of denominator is 1.)

A rational expression is **irreducible** if the numerator and the denominator have **no** common factors other than 1.

A rational expression is **reducible** if the numerator and the denominator have common factors other than 1.

Example 1 The fraction $\frac{x+1}{(x+1)(x-2)}$ is **proper** and **reducible.**

By canceling the common factor $x+1$, we obtain the proper and irreducible fraction $\frac{1}{x-2}$.

Example 2 The fraction $\frac{(x+1)(x+2)}{(x+1)(x-2)}$ is **improper and reducible**.

By canceling the common factor $x+1$, and using long division we obtain $1+\frac{4}{x-2}$, in which the fractional part is **proper a**nd **irreducible**.

Example 3 The fraction $\frac{x^2+5}{x+1}$ is **improper** and **irreducible**.

Using long division this fraction can be expressed as the sum of a polynomial quotient and a proper, irreducible fraction, Thus $\frac{x^2+5}{x+1}=x-1+\frac{6}{x+1}$.

In working with rational expressions, we shall exclude those values (called **excluded values**) of the variable in the denominator which make the denominator zero. At the excluded values, the rational expressions are undefined. We should note however that some rational expressions are defined for all real values of the independent variable: For .example, $\frac{x}{x^2+1}$ is defined for all real values of x, since the denominator is never zero.

From the above examples, our main concern in the integration of rational functions is to integrate a **proper** and **irreducible** rational fraction. We will assume that the degree of the numerator polynomial is less than the degree of the denominator polynomial and that the numerator and the denominator do not have any common linear or quadratic factors.

Classification of Integration of Rational Functions

We may classify the integration of rational functions by antiderivative type, by integrand type, or by method type.

By **antiderivative type**, **w**hen a proper irreducible rational function is integrated, the antiderivative is either a rational function, a logarithmic function, an arctangent function, or the sum of any two or all three types.

The coverage of rational functions in this book is as follows:

Rational Functions I By power rule, by natural log rule; by *u*-substitution See **Lesson 31**

Examples

1. $\int \frac{1}{x^2}dx$; **2**. $\int \frac{1}{x}dx$; **3.** $\int \frac{1}{x-2}dx$ **4.** $\int \frac{x}{x^2+2}\,dx$**;** **5.** $\int \frac{x^2}{(4x^3+2)^5}\,dx$**;**

6. $\int \frac{5x}{x^2+1}dx$**;** **7.** $\int \frac{5x}{\left(x^2+1\right)^3}dx$**; 8.** $\int \frac{3x}{x^2+4}\,dx$**;**

Rational Functions II By Partial Fractions decomposition

The antiderivative type here is a logarithmic function See **Lesson 40.**

Examples

10. $\int \frac{x-5}{x^2+x-2}\,dx$; **11..** $\int \frac{dx}{a^2-x^2}$; **12.** $\int \frac{x^2+3x+6}{x+2}\,dx$;

Rational Functions III By Trigonometric Substitution.. See **Lesson 41**

Here the antiderivative type is an arctangent function.

Examples **14**..$\int \frac{dx}{1+x^2}$; **15**. $\int \frac{dx}{a^2+x^2}$; **16,** $\int \frac{1}{1+4x^2}\,dx$; **17,** $\int \frac{dx}{(1+x^2)^2}$;

18.$\int \frac{dx}{(1+x^2)^3}$; **19.** $\int \frac{x}{x^2-6x+13}dx$; **20**. $\int \frac{1}{4x^2+6x+9}dx$; **21.** $\int \frac{dx}{a^2-x^2}$

Rational Functions I

Basic Formulas

$$\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} \text{ if } n \neq -1 \\ \\ \ln x \text{ if } n = -1 \end{cases} \qquad \textbf{(A)}$$

(Same as $\int x^n dx = \frac{x^{n+1}}{n+1}$ if $n \neq -1$; but , if $n = -1$, $\int x^n dx = \ln x$)

Below, we cover three main cases.

Case 1: Rational functions such as $\int \frac{1}{x^2}dx$, $\int \frac{1}{x^3}dx$, $\int \frac{1}{x^7}dx$ or $\int \frac{1}{x^n}dx$, $n \neq 1$. Here, we can use the **power rule** $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, the rule we used for polynomials.

Example 1 Find $\int \frac{1}{x^2}dx$

Solution

$$\int \frac{1}{x^2}dx = \int x^{-2}dx$$
$$= \frac{x^{-2+1}}{-2+1} + C$$
$$= \frac{x^{-1}}{-1} + C$$
$$= -\frac{1}{x} + C.$$

Example 2 Find $\int \frac{1}{x^3}dx$

Solution

$$\int \frac{1}{x^3}dx = \int x^{-3}dx$$
$$= \frac{x^{-3+1}}{-3+1} + C$$
$$= \frac{x^{-2}}{-2} + C$$
$$= -\frac{1}{2x^2} + C.$$

Example 3 Find $\int \frac{1}{x^7}dx$

Solution

$$\int \frac{1}{x^7}dx = \int x^{-7}dx$$
$$= \frac{x^{-7+1}}{-7+1} + C$$
$$= \frac{x^{-6}}{-6} + C$$
$$= -\frac{1}{6x^6} + C.$$

Case 2: The simple **rational functions** such as $f(x) = \frac{1}{x}$; $f(x) = \frac{1}{x-2}$

Example 4 Find $\int \frac{1}{x}dx$

We **cannot** use the power rule here since the result would be undefined.

(Note: $\int \frac{1}{x}dx = \int x^{-1}dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0} = \frac{1}{0}$, which is undefined). The impediment here is that $n = -1$ when $f(x)$ is in exponential form. So we resort to the use of logarithms. We use the lower rule in **(A)** above (p.216).

Solution $\int \frac{1}{x}dx = \ln|x| + C$ (which means $\int \frac{1}{x}dx = \ln x + C$ if $x > 0$;

$\int \frac{1}{x}dx = \ln(-x) + C$ if $x < 0$.

$\uparrow$

(The negative of or the opposite of x)

Example 5 Find $\int \frac{1}{x-2}dx$

Solution $\int \frac{1}{x-2}dx = \ln|x-2| + C$.

Case 3a: Integrating Rational Functions by Simple U-substitution Method

Rational functions such as $\int \frac{5x}{\left(x^2+1\right)^3}dx$, and $\int \frac{5x}{x^2+1}dx$ can be found using simple *u*-substitution.

Example 6 Find $\int \frac{5x}{\left(x^2+1\right)^3}dx$ <--This satisfies the condition guidelines in Lesson 30.

Solution Let $u = x^2+1$

Then $\frac{du}{dx} = 2x$ and $dx = \frac{du}{2x}$ or $du = 2xdx$

Now, replace dx in $\int \frac{5x}{\left(x^2+1\right)^3}dx$ by $\frac{du}{2x}$, and x^2+1 by u.

Then
$$\int \frac{5x}{(x^2+1)^3}\,dx = \int \frac{5x}{u^3}\cdot\frac{du}{2x}$$
$$= \int \frac{5}{u^3}\cdot\frac{du}{2} \quad \text{(canceling the common } x\text{)}$$
$$= \frac{5}{2}\int \frac{du}{u^3}$$
$$= \frac{5}{2}\int u^{-3}du$$
$$= \frac{5}{2}\cdot\frac{u^{-3+1}}{-3+1}+C$$
$$= \frac{5}{2}\cdot\frac{u^{-2}}{-2}+C$$
$$= -\frac{5}{4}\cdot u^{-2}+C$$
$$= -\frac{5}{4}\cdot\frac{1}{u^2}+C$$
$$= -\frac{5}{4(x^2+1)^2}+C \qquad (u = x^2+1)$$

Therefore, $\int \frac{5x}{\left(x^2+1\right)^3}dx = -\frac{5}{4(x^2+1)^2}+C$.

Example 7

Find $\int \frac{x}{x^2+2}\,dx$;

Step 1: Let $u = x^2 + 2$

Then $\frac{du}{dx} = 2x$ and from which $dx = \frac{du}{2x}$

Step 2: Substitute u for $x^2 + 2$; $\frac{du}{2x}$ for dx in $\int \frac{x}{x^2+2}\,dx$.

$$\text{Then } \int \frac{x}{u}\frac{du}{2x}$$

$$= \int \frac{1du}{2u}$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + c$$

Step 3: Substitute $x^2 + 2$ for u.

$$= \frac{1}{2}\ln|x^2+2| + c$$

Then $\int \frac{x}{x^2+2}\,dx = \frac{1}{2}\ln|x^2+2| + c$.

Example 8

Find $\int \frac{x}{(x^2+2)^3}\,dx$

Step 1: Let $u = x^2 + 2$

Then $\frac{du}{dx} = 2x$ and from which $dx = \frac{du}{2x}$

Step 2: Substitute u for $x^2 + 2$; $\frac{du}{2x}$ for dx in $\int \frac{x}{(x^2+2)^3}\,dx$.

$$\text{Then } \int \frac{x}{u^3}\frac{du}{2x}$$

$$= \int \frac{1du}{2u}$$

$$= \frac{1}{2}\int \frac{du}{u^3}$$

$$= \frac{1}{2}\int u^{-3}du$$

$$= \frac{1}{2} \bullet \frac{u^{-3+1}}{-3+1} + C$$

$$= \frac{1}{2} \bullet \frac{u^{-2}}{-2} + C$$

$$= -\frac{1}{4u^2} + C$$

Step 3: Substitute $x^2 + 2$ for u.

$$\text{Then} \int \frac{x}{(x^2+2)^3}\,dx = -\frac{1}{4(x^2+2)^2} + C$$

Example 9

Find $\int \frac{x^2}{\left(4x^3+7\right)^5}\,dx$

Step 1: Let $u = 4x^3 + 7$

Then $\frac{du}{dx} = 12x^2$ and

from which $dx = \frac{du}{12x^2}$

Step 2 Substitute u for

$4x^3 + 7$; $\frac{du}{12x^2}$ for dx in

$\int \frac{x^2}{\left(4x^3+2\right)^5}\,dx$ to obtain .

$$= \int \frac{x^2}{u^5} \bullet \frac{du}{12x^2}$$

$$= \frac{1}{12}\int \frac{du}{u^5}$$

$$= \frac{1}{12}\int u^{-5}du$$

Step 3: Integrate

$$= \frac{1}{12} \bullet \frac{u^{-5+1}}{-5+1} + C$$

$$= \frac{1}{12} \bullet \frac{u^{-4}}{-4} + C$$

$$= -\frac{1}{48}\frac{1}{u^4} + C$$

Step 4: $= -\frac{1}{48(4x^3+7)^4} + C$

(Substituting $4x^3 + 7$ for u)

.

Example 10

Find $\int \frac{3x}{x^2+4}\,dx$

Step 1: Let $u = x^2 + 4$

Then $\frac{du}{dx} = 2x$ and from which

$dx = \frac{du}{2x}$

Step 2: Substitute u for $x^2 + 4$;

$\frac{du}{2x}$ for dx in $\int \frac{3x}{x^2+4}\,dx$.

Then $\int \frac{3x}{u}\frac{du}{2x}$

$$= \int \frac{3du}{2u}$$

$$= \frac{3}{2}\int \frac{du}{u}$$

$$= \frac{3}{2}\ln|u| + c$$

Step 3: Substitute $x^2 + 4$ for u.

Then $\int \frac{3x}{x^2+4}\,dx = = \frac{3}{2}\ln|x^2+4| + c$

Note: Simple substitution will not work in $\int \frac{3x+2}{x^2+4}\,dx$. Why?

Extra: Which of the following can be integrated using simple *u*-substitution ?

$\int \frac{3x+2}{4x^2+1}\,dx$ or $\int \frac{3x}{4x^2+1}\,dx$?

Example 11

Find $\int \frac{2x^3+4x^2+11x+16}{x^2+4}\,dx$

Step 1: Using long division

$$\int \frac{2x^3+4x^2+11x+16}{x^2+4}\,dx$$

$$= \int 2x\,dx + \int 4dx + \int \frac{3x}{x^2+4}\,dx$$

Step 2: Integrate the first two terms and use the result of Example 10 for

$\left(\int \frac{3x}{x^2+4}\,dx = \frac{3}{2}\ln|x^2+4| + c\right)$ to obtain

$$x^2+4x+\frac{3}{2}\ln|x^2+4|+c$$

$$\therefore \int \frac{2x^3+4x^2+11x+16}{x^2+4}\,dx$$

$$= x^2+4x+\frac{3}{2}\ln|x^2+4|+c$$

Case 3b: **Rational functions** such as $\int \frac{x+6}{(x+4)^3}dx$,, and $\int \frac{x^2+5}{(x+4)^3}dx$ can be integrated using simple u-substitution, but there is an **extra step** to completely express the integral in terms of u only.

Example 12: Find $\int \frac{x+6}{(x+4)^3}dx$

Step 1: Let $u = x+4$**,** Then $\frac{du}{dx} = 1$, and $du = dx$. After substituting u for $x+4$, and du for dx in $\int \frac{x+6}{(x+4)^3}dx$, we obtain $\int \frac{x+6}{u^3}du$, and we still have the variable x, which we express in terms of u as follows:

Step 2: From $u = x+4$, $x = u-4$, and substituting for x in $\int \frac{x+6}{u^3}du$, we obtain $\int \frac{u-4+6}{u^3}du$.

$$= \int \frac{u+2}{u^3}du$$

$$= \int \frac{u}{u^3}du + \int \frac{2}{u^3}du \quad \text{(splitting the numerators)}$$

$$= \int \frac{1}{u^2}du + \int \frac{2}{u^3}du$$

$$= \int u^{-2}du + 2\int u^{-3}du$$

$$= -u^{-1} + \frac{2}{-2u^2} + C$$

$$= -\frac{1}{u} - \frac{1}{u^2} + C$$

$$\int \frac{x+6}{(x+4)^3}dx = -\frac{1}{x+4} - \frac{1}{(x+4)^2} + C.$$

Example 12: Find $\int \frac{x^2+5}{(x+4)^3}dx$.

Step 1: Let $u = x+4$**,** Then $\frac{du}{dx} = 1$, and $du = dx$. After substituting u for $x+4$, and du for dx in $\int \frac{x^2+5}{(x+4)^3}dx$, we obtain $\int \frac{x^2+5}{u^3}du$, and we still have x^2 which we express in terms of u as follows:

Step 2: From $u = x+4$, $x = u-4$, and squaring, $x^2 = u^2 - 8u + 16$

We now substitute for x^2 in $\int \frac{x^2+5}{u^3}du$, to obtain $\int \frac{u^2-8u+16+5}{u^3}du$

$$\int \frac{u^2-8u+21}{u^3}du = \int \frac{u^2}{u^3}du + \int \frac{-8u}{u^3}du + \int \frac{21}{u^3}du$$

$$= \int \frac{1}{u}du - 8\int \frac{1}{u^2}du + \int \frac{21}{u^3}du$$

Step 3 Integrate $= \ln|u| - \frac{8u^{-1}}{-1}du + \frac{21u^{-2}}{-2} + C$

$$= \ln|u| + \frac{8}{u} - \frac{21}{2u^2} + C$$

$$\int \frac{x^2+5}{(x+4)^3}dx = \ln|x+4| + \frac{8}{x+4} - \frac{21}{2(x+4)^2} + C. \quad (u = x+4)$$

31 Exercises A

1. If $n=-1$ then $\int x^n dx =$

2. If $n \neq -1$, then $\int x^n dx =$

3. Find $\int \frac{1}{x}dx$

4. Why do we write $\ln|x|+C$ instead of $\ln x + C$

5 Find $\int \frac{1}{x-2}dx$

6. Find $\int \frac{1}{x^2}dx$ (Hint: use power rule)

7. Find $\int \frac{1}{x^3}dx$

8. Find $\int \frac{5x}{(x^2+1)^3}dx$

9. Find $\int \frac{x}{x^2+2}\,dx$;

10. Find $\int \frac{x}{(x^2+2)^3}\,dx$

11. Find $\int \frac{x^2}{(4x^3+7)^5}\,dx$

12. Find $\int \frac{3x}{x^2+4}\,dx$

13, On which of the following can we simple u-substitution to integrate?

$$\int \frac{3x+2}{4x^2+1}\,dx \text{ or } \int \frac{3x}{4x^2+1}\,dx$$

14. Find $\int \frac{2x^3+4x^2+11x+16}{x^2+4}\,dx$

15. Find $\int \frac{x+6}{(x+4)^3}dx$.

16. Find $\int \frac{x^2+5}{(x+4)^3}dx$

Answers: **1.** $\ln|x|+C$; **2.** $\frac{x^{n+1}}{n+1}+C$; **3.** $\ln|x|+C$;**5.** $\ln|x-2|+C$;

6 .$-\frac{1}{x}+C$; **7.** $-\frac{1}{2x^2}+C$; **8.** $-\frac{5}{4(x^2+1)^2}+C$; **9.** $=\frac{1}{2}\ln|x^2+2|+c$;

10. $-\frac{1}{4(x^2+2)^2}+C$; **11.** $-\frac{1}{48(4x^3+7)^4}+C$; **12.** $\frac{3}{2}\ln\left(x^2+4\right)+c$

13. $\int \frac{3x}{4x^2+1}\,dx$; **14.** $x^2+4x+\frac{3}{2}\ln\left(x^2+4\right)+c$; **15,** $-\frac{1}{x+4}-\frac{1}{(x+4)^2}+C$

16. $\ln|x+4|+\frac{8}{x+4}-\frac{21}{2(x+4)^2}+C$;

Lesson 31 Exercises B

1. If $n = -1$ then $\int x^n dx =$

2. If $n \neq -1$, then $\int x^n dx =$

3. Find $\int \frac{1}{3x} dx$

4. Why do we write $\ln|x| + C$ instead of $\ln x + C$

5 Find $\int \frac{1}{x-3} dx$

6. Find $\int \frac{1}{x^3} dx$ (Hint: use power rule)

7. Find $\int \frac{1}{x^6} dx$

8. Find $\int \frac{6x}{\left(x^2-1\right)^3} dx$

9. Find $\int \frac{x^2}{x^3+2} dx$;

10. Find $\int \frac{x}{\left(x^2-2\right)^4} dx$

11. Find $\int \frac{x^3}{(5x^4+2)^4} dx$

12. Find $\int \frac{4x}{x^2+5} dx$

13, On which of the following can we simple *u*-substitution to integrate?

(a) $\int \frac{5x+2}{4x^2+1} dx$ or (b) $\int \frac{5x}{4x^2+1} dx$

14. Find $\int \frac{x^2+1}{(x-1)^3} dx$

Answers: **3.** $\frac{1}{3}\ln|x| + C$; **5.** $\ln|x-3| + C$ **6.** $-\frac{1}{2x^2} + C$; **7.** $-\frac{1}{5x^5} + C$;

8. $-\frac{3}{2(x^2-1)^2} + C$; **9.** $\frac{1}{3}\ln\left|x^3+2\right| + C$; **10.** $-\frac{1}{6(x^2-2)^3} + C$;

11. $-\frac{1}{60(5x^4+2)^3} + C$; **12.** $2\ln\left(x^2+5\right) + C$; **13.** b.;

14. $\ln|x-1| - 2(x-1)^{-1} - (x-1)^{-2} + C$. or

$$\ln|x-1| - \frac{2}{x-1} - \frac{1}{(x-1)^2} + C$$

Lesson 32

Integration of Radical Functions I

Classification of Integration of Radical Functions

We may classify the integration of radical functions by antiderivative type, by integrand type, or by method type. For some functions ,we will use simple *u*-substitution, and for others we will use trigonometric substitution. In this lesson, we use only simple *u*-substitution. In Lesson 43, Integration of Radical Functions II, we use trigonometric substitution. Before proceeding, review Lesson 30.

Integration of Radical Functions Using Simple U-Substitution

Examples **1.** $\int \sqrt{x-5}\,dx$**;** **2.** $\int \frac{4x^2}{\sqrt{x^3+1}}\,dx$**;** **3.** $\int 4x^2\sqrt{x^3+1}\,dx$**.**

Note: **1** is composite; **2** and **3** contain composite functions.

Example 1 Find $\int \sqrt{x-5}\,dx$

Solution We use simple **u-substitution** with **two approaches**
Both approaches rationalize the integrand..

Approach 1

$\int \sqrt{x-5}\,dx$

Let $u = x-5$

Then $\frac{du}{dx} = 1$, and $du = dx$

Substituting for $x-5$ and dx in

$\int \sqrt{x-5}\,dx$, we obtain

$\int \sqrt{u}\,du$

$= \int u^{\frac{1}{2}}du$

$= \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$= \frac{2}{3}u^{\frac{3}{2}} + C$

$= \frac{2}{3}(x-5)^{\frac{3}{2}} + C$ (back to x).

$= \frac{2}{3}\left(\sqrt{x-5}\right)^3 + C$ **.** or

$= \frac{2}{3}(x-5)\sqrt{x-5} + C$

Approach 2

Let $u = \sqrt{x-5}$ (A)

Then $u^2 = x-5$

Using implicit differentiation,

$2u\frac{du}{dx} = 1$, and $2u\,du = dx$

Substituting for $\sqrt{x-5}$ and dx in

$\int \sqrt{x-5}\,dx$, we obtain

$\int u \bullet 2u\,du$

$= \int 2u^2du$

$= 2\int u^2du$

$= 2 \bullet \frac{u^3}{3} + C$

$= \frac{2}{3}\left(\sqrt{x-5}\right)^3 + C$ (back to x). or

$= \frac{2}{3}(x-5)\sqrt{x-5} + C$

Example 2 Find $\int \frac{4x^2}{\sqrt{x^3+1}}\,dx$

We use simple **u-substitution** with t**wo approaches.**

Approach 1

Step 1: Let $u = x^3 + 1$.

Then $\frac{du}{dx} = 3x^2$ and from which

$$dx = \frac{du}{3x^2}$$

Step 2: Now, replace dx in $\int \frac{4x^2}{\sqrt{x^3+1}}\,dx$ by $\frac{du}{3x^2}$, and x^3+1 by u. Then ,

$$\int \frac{4x^2}{\sqrt{x^3+1}}\,dx$$

$$= \int \frac{4x^2}{\sqrt{u}} \cdot \frac{du}{3x^2}$$

$$= \int \frac{4}{u^{\frac{1}{2}}} \cdot \frac{du}{3} \text{ (canceling the common } x^2\text{)}$$

$$= \frac{4}{3}\int \frac{1}{u^{\frac{1}{2}}}\,du$$

$$= \frac{4}{3}\int u^{-\frac{1}{2}}du$$

$$= \frac{4}{3} \cdot \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \text{ (using power rule)}$$

$$= \frac{4}{3} \cdot \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{8}{3}u^{\frac{1}{2}} + C$$

$$= \frac{8}{3}\sqrt{u} + C$$

$$= \frac{8}{3}\sqrt{x^3+1} + C \text{ (replacing } u \text{ by } x^3\ +1$$

$$\int \frac{4x^2}{\sqrt{x^3+1}}\,dx = \frac{8}{3}\sqrt{x^3+1} + C\ .$$

Note above that $\sqrt{x^3+1}$, is composite.

Approach 2

Step 1: **Rationalize the integrand**

Let $u = \sqrt{x^3+1}$.

Then $u^2 = x^3 + 1$

Using implicit differentiation,

$2u\frac{du}{dx} = 3x^2$, and from which

$$dx = \frac{2udu}{3x^2}$$

Step 2: Substitute u for $\sqrt{x^3+1}$; $\frac{2udu}{3x^2}$ for dx in $\int \frac{4x^2}{\sqrt{x^3+1}}\,dx$ to obtain

$$\int \frac{4x^2}{u} \bullet \frac{2udu}{3x^2}$$

$$= \frac{8}{3}\int 1du$$

$$= \frac{8}{3}u + c$$

$$= \frac{8}{3}\sqrt{x^3+1} + c \quad (u = \sqrt{x^3+1}).$$

.

Example 3. Find $\int 4x^2\sqrt{x^3+1}\;dx$

Solution : We use simple **u-substitution** with **two approaches**

Approach 1 .

Step 1 Let $u = x^3+1$.

Then $\frac{du}{dx} = 3x^2$ and from which $dx = \frac{du}{3x^2}$

Step 2: Substitute u for x^3+1; and $\frac{du}{3x^2}$ for dx in $\int 4x^2\sqrt{x^3+1}\;dx$. Then

$$\int 4x^2\sqrt{x^3+1}\;dx = \int 4x^2u^{\frac{1}{2}} \cdot \frac{du}{3x^2}$$

$$= \frac{4}{3}\int u^{\frac{1}{2}}du$$

$$= \frac{4}{3}\int \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\;du$$

$$= \frac{4}{3}\left(\frac{2}{3}u^{\frac{3}{2}}\right)+C$$

$$= \frac{8}{9}\left(u^{\frac{3}{2}}\right)+C$$

$$= \frac{8}{9}(x^3+1)^{\frac{3}{2}}+C$$

$$= \frac{8}{9}\left(\sqrt{x^3+1}\right)^3+C$$

$$\int 4x^2\sqrt{x^3+1}\;dx = \frac{8}{9}\left(\sqrt{x^3+1}\right)^3+C.$$

$$= \frac{8}{9}(x^3+1)\sqrt{x^3+1}+C$$

Note above that $\sqrt{x^3+1}$, is composite with the "inside" function being x^3+1.

Approach 2

Step 1: Rationalize the integrand

Let $u = \sqrt{x^3+1}$.

Then $u^2 = x^3+1$

Then $2u\frac{du}{dx} = 3x^2$ and from which

$$dx = \frac{2udu}{3x^2}$$

Step 2: Substitute u for $\sqrt{x^3+1}$; $\frac{2udu}{3x^2}$ for dx in $\int 4x^2\sqrt{x^3+1}\;dx$ to obtain

$$\int 4x^2u \bullet \frac{2udu}{3x^2}$$

$$= \frac{8}{3}\int u^2du$$

$$= \frac{8}{3} \bullet \frac{u^3}{3}+c$$

$$= \frac{8}{9}u^3+c$$

$$= \frac{8}{9}\left(\sqrt{x^3+1}\right)^3+c \quad (u = \sqrt{x^3+1}).$$

$$\text{or } = \frac{8}{9}(x^3+1)\sqrt{x^3+1}+C$$

Example 4 Find $\int \frac{dx}{1+\sqrt{x}}$

Solution: We use simple ***u*-substitution** with **two approaches**
Both approaches rationalize the integrand.

Approach 1

$$\int \frac{dx}{1+\sqrt{x}}$$

Let $\sqrt{x} = u$

Then $x = u^2$

$\frac{dx}{du} = 2u$, or $2u\frac{du}{dx} = 1$

and from either,

$dx = 2udu$

Substitute accordingly.

Then $\int \frac{dx}{1+\sqrt{x}} = \int \frac{2udu}{1+u}$

$= 2\int \frac{u}{1+u}du$

$= 2\int\left(1 - \frac{1}{1+u}\right)du$

$= 2\left(\int 1du - \int \frac{1}{1+u}du\right)$

$= 2[u - \ln(u+1)] + C$

$= 2u - 2\ln(u+1)] + C$

$= 2\sqrt{x} - 2\ln(\sqrt{x}+1) + C$

$(\sqrt{x} = u)$

Approach 2

$$\int \frac{dx}{1+\sqrt{x}}$$

Step 1: Let $\sqrt{x} = u^2$

Then $x = u^4$

$4u^3\frac{du}{dx} = 1$, and $dx = 4u^3du$

Substitute accordingly.

Then $\int \frac{dx}{1+\sqrt{x}}$

$= \int \frac{4u^3du}{1+u^2}$

$= 4\int \frac{u^3}{1+u^2}du$

Step 2: $= 4\int\left(u - \frac{u}{1+u^2}\right)du$

$= 4\left[\int udu - \int \frac{u}{1+u^2}du\right]$ (2nd term by simple-subst.)

$= 4\left[\frac{u^2}{2} - \frac{1}{2}\ln(u^2+1)\right] + C$ (by simple-substitution).

$= 2u^2 - 2\ln(u^2+1)] + C$

$= 2\sqrt{x} - 2\ln(\sqrt{x}+1) + C$ $\quad (\sqrt{x} = u^2)$

Note: In Approach 2, Step 2:

We used simple *t*-substitution to find $\int \frac{u}{1+u^2}du$ as follows: **:** Let $t = u^2 + 1$,

then $\frac{dt}{du} = 2u$, and $du = \frac{dt}{2u}$

$\int \frac{udt}{2ut} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln t = \frac{1}{2}\ln(1+u^2)$.

Example 5 Find $\int \frac{\sqrt{1+x}}{1-x}\,dx$

Solution We use simple u-substitution

Step 1: Rationalize the integrand

Let $u = \sqrt{1+x}$

Then $u^2 = 1+x$ (A)

Now, we find an expression for $1-x$ in terms of u.

From (A), $-u^2 = -(1+x)$ (First multiply both sides of (A) by -1

$2-u^2 = -1-x+2$ (Add 2 to both sides of the equation)

$2-u^2 = 1-x$

$x = u^2 - 1$ (solving for x)

$\frac{dx}{du} = 2u$ or implicitly $, -2u\frac{du}{dx} = -1$

$dx = 2udu$

Step 2: Substitute u for $\sqrt{1+x}$; $2udu$ for dx in $\int \frac{\sqrt{1+x}}{1-x}\,dx$

$$\int \frac{\sqrt{1+x}}{1-x}\,dx = \int \frac{u}{2-u^2} \bullet \frac{2udu}{1}$$

$$= 2\int \frac{u^2du}{2-u^2}$$

$$= -2\int \frac{u^2du}{u^2-2}$$

$$= -2\left(\int 1du + \int \frac{2du}{u^2-2}\right) \quad \text{(by division)}$$

$$= -2\int du - 4\int \frac{du}{u^2-2}$$

$$= -2u - 4\int \frac{du}{u^2-2}$$

$$= -2u - 4\left[\int -\frac{\sqrt{2}}{4} \bullet \frac{1du}{u+\sqrt{2}} + \int \frac{\sqrt{2}}{4} \bullet \frac{1du}{u-\sqrt{2}}\right]$$

$$= -2u + \sqrt{2}\int \frac{1du}{u+\sqrt{2}} - \sqrt{2}\int \frac{1du}{u-\sqrt{2}}$$

$$= -2u + \sqrt{2}\ln\left|u+\sqrt{2}\right| - \sqrt{2}\ln\left|u-\sqrt{2}\right| + C$$

For $\int \frac{1}{u^2-2}du$:

We use partial fraction decomposition to integrate. See Rational Functions II (Lesson 40) to review partial fraction decomposition from Pre-Calculus)

$$= -2\sqrt{1+x} + \sqrt{2}\ln\left|\sqrt{1+x}+\sqrt{2}\right| - \sqrt{2}\ln\left|\sqrt{1+x}-\sqrt{2}\right| + C$$

$$= -2\sqrt{1+x} + \sqrt{2}\ln\left|\frac{\sqrt{1+x}+\sqrt{2}}{\sqrt{1+x}-\sqrt{2}}\right| + C.$$

Example 6 Find $\int \sqrt[3]{2x-3}\; x\, dx$

Solution We use simple u-substitution.

Step 1: Rationalize the integrand.

Let $u = \sqrt[3]{2x-3}$ or $u = (2x-3)^{\frac{1}{3}}$

$u^3 = 2x-3$

$3u^2 \dfrac{du}{dx} = 2$ and $dx = \dfrac{3u^2}{2} du$ (by implicit differentiation)

Also, $x = \dfrac{u^3+3}{2}$ (we need this for the factor "x" in $\int \sqrt[3]{2x-3}\; x\, dx$

Step 2: Substitute u for $\sqrt[3]{2x-3}$; $\dfrac{3u^2}{2} du$ for dx in $\int \sqrt[3]{2x-3}\; x\, dx$**.**

$$
\begin{aligned}
\text{Then } \int \sqrt[3]{2x-3}\; x\, dx &= \int \frac{u(u^3+3)}{2} \bullet \frac{3u^2}{2} du \\
&= \frac{3}{4}\left(\int u^3(u^3+3)\right) du \\
&= \frac{3}{4}\left(\int u^6 du + \int 3u^3 du\right) \\
&= \frac{3}{4}\left(\frac{u^7}{7} + \frac{3u^4}{4}\right) + C \\
&= \frac{3u^7}{28} + \frac{9u^4}{16} + C
\end{aligned}
$$

Step 3: We change back to the variable x. ($u = \sqrt[3]{2x-3}$)

$$
\begin{aligned}
&= \frac{3(\sqrt[3]{2x-3})^7}{28} + \frac{9(\sqrt[3]{2x-3})^4}{16} + C \\
&= \frac{3(2x-3)^{\frac{7}{3}}}{28} + \frac{9(2x-3)^{\frac{4}{3}}}{16} + C \\
&= (2x-3)^{\frac{4}{3}}\left[\frac{3(2x-3)}{28} + \frac{9}{16}\right] + C \\
&= (2x-3)^{\frac{4}{3}}\left[\frac{6x-9}{28} + \frac{9}{16}\right] + C \\
&= (2x-3)^{\frac{4}{3}}\left[\frac{4(6x-9)+7(9)}{112}\right] + C \\
&= (2x-3)^{\frac{4}{3}}\left[\frac{24x+27}{112}\right] + C \\
&= \frac{3(8x+9)}{112}(2x-3)^{\frac{4}{3}} + C \\
&= \frac{3}{112}(8x+9)(2x-3)^{\frac{4}{3}} + C \\
&= \frac{3}{112}(8x+9)(2x-3)\sqrt[3]{2x-3} + C
\end{aligned}
$$

$$\int \sqrt[3]{2x-3}\; x\, dx = \frac{3}{112}(16x^2-6x-27)(\sqrt[3]{2x-3}) + C.$$

Example 7 Find $\int \frac{x^2+4}{\sqrt{x+4}}dx$ (This example is similar to Case 3b of Lesson 31).

Step 1 Let $u = x+4$**,** Then $\frac{du}{dx} = 1$, and $du = dx$. After substituting u for $x+4$, and du for dx in $\int \frac{x^2+4}{\sqrt{x+4}}dx$, we obtain $\int \frac{x^2+4}{u^{\frac{1}{2}}}du$, and we still have x^2 which we express in terms of u as follows:

Step 2: From $u = x+4$, $x = u-4$, and squaring,

$$x^2 = u^2 - 8u + 16$$

We now substitute for x^2 in $\int \frac{x^2+4}{u^{\frac{1}{2}}}du$, to obtain

$$\int \frac{u^2 - 8u + 16 + 4}{u^{\frac{1}{2}}}du$$

$$= \int \frac{u^2}{u^{\frac{1}{2}}}du + \int \frac{-8u}{u^{\frac{1}{2}}} + \int \frac{20}{u^{\frac{1}{2}}}$$

$$= \int u^{\frac{3}{2}}du - \int 8u^{\frac{1}{2}}du + \int 20u^{-\frac{1}{2}}du$$

Step 3: Integrate now.

$$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}\left(8u^{\frac{3}{2}}\right) + \frac{2}{1}\left(20u^{\frac{1}{2}}\right) + C$$

$$= u^{\frac{1}{2}}\left(\frac{2}{5}u^2 - \frac{16}{3}u + 40\right) + C$$

$$= (x+4)^{\frac{1}{2}}\left[\frac{2}{5}(x+4)^2 - \frac{16}{3}(x+4) + 40\right] + C$$

$$= (x+4)^{\frac{1}{2}}\left[\frac{6x^2+48x+96}{15} - \frac{80(x+4)}{15} + \frac{600}{15}\right] + C$$

$$\int \frac{x^2+4}{\sqrt{x+4}}dx = \frac{2}{15}\sqrt{x+4}\left[3x^2 - 16x + 188\right] + C\textbf{.}$$

Lesson 32 Exercises A

1. Find $\int \sqrt{x-5}\, dx$

2. Find the $\int \frac{4x^2}{\sqrt{x^3+1}}\, dx$

3. Find $\int 4x^2\sqrt{x^3+1}\, dx$

4. Find $\int \frac{dx}{1+\sqrt{x}}$

.

5. Find $\int \sqrt[3]{2x-3}\; x\, dx$

6. Find $\int \frac{\sqrt{1+x}}{1-x}\, dx$

7. Find $\int \frac{x^2+4}{\sqrt{x+4}} dx$

Answers: 1. $\frac{2}{3}\left(\sqrt{x-5}\right)^3+C$ or $\frac{2}{3}(x-5)\sqrt{x-5}+C$**;** **2.** $\frac{8}{3}\sqrt{x^3+1}+C$;

3. $\frac{8}{9}\left(\sqrt{x^3+1}\right)^3+c$; **4.** $2\sqrt{x}-2\ln(\sqrt{x}+1)+C$**;**

5. $\frac{3}{112}(16x^2-6x-27)(\sqrt[3]{2x-3})$**;** **6.** $-2\sqrt{1+x}+\sqrt{2}\ln\left|\frac{\sqrt{1+x}+\sqrt{2}}{\sqrt{1+x}-\sqrt{2}}\right|+C$**;**

7 . $\frac{2}{15}\sqrt{x+4}\left[3x^2-16x+188\right]+C$

Lesson 32 Exercises B

1. Find $\int \sqrt{x-3}\, dx$

2. Find the $\int \frac{4x^3}{\sqrt{x^4+2}}\, dx$

3. Find $\int 4x^4\sqrt{x^5+3}\, dx$

4. Find $\int \frac{dx}{1+\sqrt{x}}$

5. Find $\int \sqrt[3]{3x-2}\; x\, dx$

6. Find $\int \frac{\sqrt{2+x}}{2-x}\, dx$

7. Find $\int \frac{\sqrt{x}}{\sqrt[3]{x+2}}\, dx$

Answers: 1. $\frac{2}{3}\left(\sqrt{x-3}\right)^3+C$ or $\frac{2}{3}(x-3)\sqrt{x-3}+C$**;** **2.** $2\sqrt{x^4+2}$;

3. $\frac{8}{15}\left(\sqrt{x^5+3}\right)^3+C$ or $\frac{8}{15}(x^5+3)\sqrt{x^5+3}+C$

4. $2\sqrt{x}-2\ln(\sqrt{x}+1)+C$ **;**

5. $\frac{1}{14}(2x+1)\left(\sqrt[3]{3x-2}\right)^4+C$; or $\frac{1}{14}\left(6x^2-x-2\right)\left(\sqrt[3]{3x-2}\right)+C$

6. $-2\sqrt{2+x}+2\ln\left|\frac{2+\sqrt{2+x}}{-2+\sqrt{2+x}}\right|$

7. $\frac{6}{7}x^{\frac{7}{6}}-\frac{12}{5}x^{\frac{5}{6}}+8\sqrt{x}-48x^{\frac{1}{6}}+48\sqrt{2}\mathrm{Tan}^{-1}(\frac{1}{2}x^{\frac{1}{6}}\sqrt{2})+C$.

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Power of Ratios

CHAPTER 6

Applications of Ratios and Proportion in Physics and Chemistry

Lesson 16

Boyle's Law

(An inverse proportion relationship)

Preliminaries:
Pressure = Force per unit area
Postulate: The pressure of a gas is the result of collisions of the molecules of the gas with the walls of the container.
Boyle's Law deals with the effects of pressure changes on the volume of a gas at constant temperature.

Boyle's Law:
Boyle's Law states that at constant temperature, the pressure exerted on a fixed quantity of a gas is inversely proportional to the volume of the gas. (i.e., pressure and volume vary indirectly).

The pressure being inversely proportional to the volume implies that when the volume increases, the pressure decreases, and when the volume decreases, the pressure increases.
Note: The product of pressure, P, and volume, V, is a constant. If the pressure decreases, the volume must increase the keep the product PV constant.

Boyle's Law in equation form

Let the initial pressure on a gas = P_1; and let the initial volume of the gas = V_1
Let the final pressure of the gas = P_2; and let the final volume of the gas = V_2

Then Boyle's Law states that $P_1V_1 = P_2V_2$ (Note: **product = product**, inverse proportion)

Example When the pressure on a given mass of a gas is 740 mm Hg, the volume is 4.7 L. Find the volume of the gas when the pressure is 720 mm Hg, assuming constant temperature.

Solution $P_1 = 740$ mm Hg, $V_1 = 4.7$ L
$P_2 = 720$ mm Hg, $V_2 = ?$

We are to find V_2. Substitute the known values in $P_1V_1 = P_2V_2$ (Boyle's Law)

$$740 \text{ mm Hg} \times 4.7 \text{ L} = 720 \text{ mm Hg} \times V_2$$

$$\frac{740 \text{ mm Hg} \times 4.7 \text{ L}}{720 \text{ mm Hg}} = V_2$$

$$4.83 \text{ L} = V_2$$

$$4.8 \text{ L} = V_2$$

$$V_2 = 4.8 \text{ L}.$$

When the pressure is 720 mm Hg, the volume is 4.8 L

Note above that we can also use any of the other methods (already covered) for solving inverse proportion problems. See Lesson 6.

Lesson 17
Charles' Law
(A direct proportion relationship)

Charles' Law deals with the effects of temperature changes on the volume of a gas. keeping the pressure constant.

Charles' Law: Charles' Law states that at constant pressure, the volume of a gas is directly proportional the temperature of the gas.
(i.e., volume and temperature vary directly)

The volume being directly proportional to the temperature implies that when the temperature increases, the volume increases, and when the temperature decreases, the volume also decreases.

Charles' Law in equation form

Let the initial volume of a gas = V_1; and let the initial temperature of the gas = T_1

Let the final volume of the gas = V_2; and let the final temperature of the gas = T_2

(Note: T_1 and T_2 are in degrees Kelvin)

Then Charles' Law states that $\frac{V_1}{T_1} = \frac{V_2}{T_2}$.(Note: **quotient = quotient**, a direct proportion)

Example At constant pressure, when the volume of a given mass of a gas is 4.2 L, the temperature is 36°. Find the volume when the temperature is 24° C.

Solution

$$\begin{aligned} V_1 &= 4.2L \\ T_1 &= 36° \\ &= (36 + 273)°K \\ T_1 &= 309°K \\ T_2 &= 24° \\ &= (24 + 273)°K \\ T_2 &= 297°K \\ V_2 &= ? \end{aligned}$$

We are to find V_2.

Substituting the above values in $\frac{V_1}{T_1} = \frac{V_2}{T_2}$

$$\begin{aligned} \frac{4.2 \text{ L}}{309°} &= \frac{V_2}{297°K} \\ 309° \times V_2 &= 4.2 \text{ L} \times 297°\text{K} \\ \text{V}_2 &= \frac{4.2 \text{ L} \times 297°K}{309°\text{K}} \\ &= 4.036 \text{ L} \\ V_2 &= 4.0 \text{ L} \end{aligned}$$

When the temperature is 24° C, the volume is 4.0 L.
Similarly, as it is in the case of Boyle's Law, note that we can also use any of the other methods (already covered) for solving direct proportion problems. See Lesson 5.
..

Lesson 18
Gay-Lussac's Law
(A direct proportion relationship)

Gay-Lussac's Law deals with the effects of temperature changes on the pressure on a gas.

Gay-Lussac's Law:
Gay-Lussac's Law states that at constant volume, the pressure on a gas is directly proportional the temperature of the gas. (That is, the pressure and temperature vary directly)

The pressure being directly proportional to the temperature implies that when the temperature increases, the pressure also increases, and when the temperature decreases, the pressure also decreases.

Gay-Lussac's Law in equation form

Let the initial pressure on a gas = P_1; and let the initial temperature of the gas = T_1

Let the final pressure on the gas = P_2; and let the final temperature of the gas = T_2

Then Gay-Lussac's Law states that $\frac{P_1}{T_1} = \frac{P_2}{T_2}$ (Note: **quotient = quotient,** a direct proportion)

Example At constant volume, when the pressure on a given mass of a gas is 720 mm Hg, the temperature is 30°C. What is the pressure when the temperature is 36°C?

Solution

$$\begin{aligned} P_1 &= 720 \text{ mm Hg}; \\ T_1 &= 30° \\ &= (30 + 273)°K; \\ T_1 &= 303°K \\ P_2 &= ? \\ T_2 &= 36°C \\ &= (36 + 273)°K \\ T_2 &= 309°K \end{aligned}$$

Substituting the above values in $\frac{P_1}{T_1} = \frac{P_2}{T_2}$

$$\frac{720 \text{ mm Hg}}{303°\text{K}} = \frac{P_2}{309°K}$$

$$303°K \times P_2 = 720 \text{ mm Hg} \times 309°\text{K}$$

$$\text{P}_2 = \frac{720 \text{ mm Hg} \times 309°\text{K}}{303°\text{K}}$$

$$\begin{aligned} P_2 &= 734.257 \text{ mm Hg} \\ &= 734 \text{ mm Hg} \end{aligned}$$

When the temperature is 36°C, the pressure is 734 mm Hg.

Note above that we can also use any of the other methods (already covered) for solving direct proportion problems. See Lesson 5.

Lesson 19
Combined Gas Law
(A combined proportion relationship)

The combined gas law is obtained from Boyle's Law, Charles' Law, and Gay-Lussac's Law.

In the combined gas law, pressure, volume, and temperature, all vary at the same time and none is constant.

The combined gas law states that the pressure on a given mass of a gas is inversely proportional to the volume and directly proportional to the temperature.

Combined Gas Law in equation form

Let the initial pressure = P_1, initial volume = V_1, and initial temperature = T_1
Let the final pressure = P_2, final volume = V_2, and final temperature = T_2

Then the combined gas Law states that $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$

Example When the pressure on a given mass of a gas is 720 mm Hg, and the volume is 6.2 L, the temperature is 28 °C. Find the temperature when the pressure is 740 mm Hg and the volume is 8.5 L
Solution Method 1

$$P_1 = 720 \text{ mm Hg},\ V_1 = 6.2 \text{ L}$$
$$T_1 = 28°C$$
$$= (28 + 273)°K$$
$$T_1 = 301°K$$
$$P_2 = 740 \text{ mm Hg}.\ V_2 = 8.5 \text{ L}$$
$$\mathrm{T}_2 = ?$$

We are to find T_2.

Substituting in $\frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2}$

$$\frac{720 \text{ mm Hg} \times 6.2 \text{ L}}{301°\text{K}} = \frac{740 \text{ mm Hg} \times 8.5 \text{ L}}{T_2}$$

$$720 \text{ mm Hg} \times 6.2 \text{ L} \times \mathrm{T}_2 = 740 \text{ mm Hg} \times 8.5 \text{ L} \times 301°\text{K}$$

$$\mathrm{T}_2 = \frac{740 \text{ mm Hg} \times 8.5 \text{ L} \times 301°\text{K}}{720 \text{ mm Hg} \times 6.2 \text{ L}}$$
$$= 424.124°K$$
$$= (424.124 - 273)°C$$
$$= 151.124$$
$$= 151°C$$

When the pressure is 740 mm Hg, and the volume is 8.5 L, the temperature is 151°C.

Note above that we can also use any of the other methods (already covered) for solving combined proportion problems. See Lesson 7.

Lesson 20

Application of Ratios in Physics-B

Velocity

If a body travels a distance, *s,* in a straight line, in time *t,* the velocity, v ,is given by the **ratio,**

$$v = \frac{s}{t}$$

Acceleration

If a body starts with a velocity v_0 and changes to velocity v_f after a time interval *t,* the body's acceleration, *a,* is given by the **ratio,** $\frac{\text{change in velocity}}{\text{time interval}}$

$$\text{Thus } a = \frac{v_f - v_0}{t}$$

Newton's second law

From Newton's second law, if a net force *F* acts on a body of mass, m, the body's acceleration, *a* is given by **the ratio,** $\frac{\text{net force}}{\text{mass}}$

$$\text{Thus } a = \frac{F}{m}.$$

Friction

If the normal force of reaction holding two surfaces together is N, and the friction force is F, the coefficient of friction, μ, for the two surfaces is given by the **ratio** $\frac{F}{N}$.

$$\text{Thus } \mu = \frac{F}{N}.$$

Circular motion

The centripetal acceleration, a_c, of a body in uniform circular motion is given by the **ratio,** $\frac{v^2}{r}$, where *r* is the radius of its path, and. v is the tangential velocity..

$$\text{Thus } a_c = \frac{v^2}{r}$$

Power

Power is the rate of doing work. If the work performed is *W*, and the time interval is *t,* then power, *P,* is given the **ratio** $\frac{W}{t}$**.** Thus $P = \frac{W}{t}$**.**

Elasticity

Modulus of elasticity is defined as the **ratio,** $\frac{\text{stress}}{\text{strain}}$

Pressure:

If a force. *F*, acts perpendicular to a surface of area, *A,* the pressure, *P,* exerted on the surface is the **ratio** $\frac{\text{force}}{\text{area}}$.. Thus $P = \frac{F}{A}$

Mechanical Advantage

The mechanical advantage of a hydraulic press is given by the **ratio** $\frac{A_{\text{out}}}{A_{\text{in}}}$**,** where A_{out} is the area of the output piston, and A_{in} is the area of the input piston.

Kinematic Viscosity

kinematic coefficient of viscosity, $\nu = \frac{\mu}{\rho}$**,** where μ = absolute viscosity and ρ = mass density

Electric field

*I*f a charge *q* at a given point is acted upon by a given force *F,* then the electric field, E, at the given point is defined as the **ratio** $\frac{\text{force}}{\text{charge}}$.

Thus $E = \frac{F}{q}$

Coulomb's Law

The force, F one charge exerts on another charge is given by

$F = k\frac{q_1 q_2}{r^2}$, where q_1, q_2 are the charges, and *r* is the distance between the charges.

Ohm's law

If the potential difference between the ends of a conductor is *V,* and the resistance of the conductor is *R,* then the current, *I,* in the conductor is given by the r**atio,** $\frac{V}{R}$.

Thus $I = \frac{\mathrm{V}}{\mathrm{R}}$.

Resistance

The resistance, *R*, of a conductor is the **ratio** $\frac{V}{I}$**,** where *V* is the potential difference across the conductor, and *I* is the current in the conductor.

Electric Current

If a quantity of charge, *q,* passes a given point in a conductor in the time interval *t,* the electric current, *I* , in the conductor is given by the **ratio,** $\frac{\text{charge}}{\text{time interval}}$

Thus $I = \frac{\mathrm{q}}{\mathrm{t}}$.

Capacitance

If the potential difference between the plates of a capacitor V, and the charge on either plates Q, then the capacitance, C, of the capacitor is the **ratio** $\frac{\text{charge}}{\text{potential difference}}$. Thus $C = \frac{Q}{V}$.

Magnetic Intensity

Magnetic intensity **H**, is defined as the **ratio** $\frac{\mathbf{B}}{\mu}$

where **B** is the magnetic field and μ is the permeability of medium

Thus $\mathbf{H} = \frac{\mathbf{B}}{\mu}$

Index of refraction The index of refraction, n, in a transparent medium is the ratio $\frac{c}{v}$, where c is the velocity of light in free space, and v is the velocity of light in the medium.

Thus $n = \frac{c}{v}$

Index of refraction For a ray of light passing from medium A into medium B at an oblique angle, the index of refraction, n, is defined by the ratio $\frac{\text{velocity of light in A}}{\text{velocity of light in B}}$

Thus $n = \frac{\text{velocity of light in A}}{\text{velocity of light in B}}$

Illumination

The illumination E, of a surface of area A, is the ratio $\frac{F}{A}$, where F is the luminous flux.

Linear Magnification

The linear magnification of an optical system is the ratio, $\frac{\text{image height}}{\text{object height}}$

Snell's law

$\mu = \frac{\sin i}{\sin r}$, where $\mu =$ refractive index, $i =$ angle of incidence, $r =$ angle of refraction

Telescope

$M = \frac{f_o}{f_e}$, where $M =$ magnifying power of objective at infinity, $f_o =$ focal length of objective,

$f_e =$ focal length of eyepiece.

Lesson 21

Application of Ratios in Chemistry

Law of definite composition; Law of multiple proportions; Valence; Writing formulas; Balancing equations, Vapor density; Specific gravity, Molecular weight of gases, Hydrogen equivalent; Problems based on equations

Law of Definite composition

The law of definite composition states that when elements combine to form compounds, the masses of the elements involved in the combination are in a definite **ratio.**

or

The law of definite composition states that in a pure compound, the masses of the elements are always in definite **proportions.**

Example In water (H_2O) the **ratio** of the mass of hydrogen to the mass of oxygen is 1 to 8, irrespective of the source of the water.

Law of Multiple Proportions

When two elements A and B combine to form more than one compound, the masses of B which combine with a fixed mass of A are in the **ratio** of small whole numbers.

Illustration of the law of multiple proportions

Nitrogen and oxygen can combine to form two different substances, nitric oxide, and nitrogen pentoxide:

Composition ⇓	Fixed Mass of Nitrogen (from experiment) ⇓	**Mass of Oxygen** (from experiment) ⇓	**Ratio** of oxygen in different compounds ⇓
Nitric Oxide	28 g of nitrogen	32 g of oxygen	32 to 64 or 1 to 2
Nitrogen pentoxide	28 g of nitrogen	64 g of oxygen	

From the above table, the masses of oxygen, 32 g and 64 g, which combine with a fixed mass of nitrogen, namely 28 g are in the **ratio** 32 to 64 or 1 to 2 (small whole numbers).

Density: The density of a substance is the **ratio** $\frac{\text{mass of substance}}{\text{volume of substance}}$

Specific Gravity: For **solids** and **liquids**, specific gravity is the **ratio** $\frac{\text{density of substance}}{\text{density of water}}$

For **gases,** specific gravity is the **ratio** $\frac{\text{density of gas}}{\text{density of air (or other gases)}}$

Vapor density: The vapor density of a gas is defined as the **ratio** $\frac{\text{weight of any volume of gas}}{\text{weight of an equal volume of air}}$

Molecule: A molecule is a group of atoms bound together such that they behave as a single particle.

Molecular weight is the weight of one molecule of a substance.
Gram-molecular weight (GMW) = Sum of the atomic weights in a molecule of the substance.

Mole
1 gram mole contains 6.02×10^{23} molecules.
1 g mole = $\frac{\text{mass in g}}{\text{molecular weight}}$

Question How many grams of $C_6H_{12}O_6$ are in one molecule of $C_6H_{12}O_6$?

Solution We use the "Units Label Method" of solving proportion problems.
Note 1: 1 mole of $C_6H_{12}O_6$ ~ 180 g of $C_6H_{12}O_6$
Note 2: 1 mole of $C_6H_{12}O_6$ contains 6.02×10^{23} molecules of $C_6H_{12}O_6$

$$1 \text{ molecule } C_6H_{12}O_6 \times \frac{1\text{mole of } C_6H_{12}O_6}{6.02 \times 10^{23} \text{molecules of } C_6H_{12}O_6} \times \frac{180 \text{ g } C_6H_{12}O_6}{1 \text{ mole } C_6H_{12}O_6}$$

$$= \cancel{1 \text{ molecule } C_6H_{12}O_6} \times \frac{\cancel{1\text{mole of } C_6H_{12}O_6}}{6.02\times 10^{23}\, \cancel{\text{molecules of } C_6H_{12}O_6}} \times \frac{180 \text{ g } C_6H_{12}O_6}{\cancel{1 \text{ mole } C_6H_{12}O_6}}$$

$$= 2.99 \times 10^{-22} \text{ g}$$

$$= 3.0 \times 10^{-22}.$$

Mole fraction: Mole fraction is the ratio $\frac{\text{moles of substance}}{\text{total number of moles}}$

Calculations based on Chemical reactions
Chemical Equations

The coefficients of a chemical equation indicate the **ratio** in which the moles of one substance reacts with the moles of another.
For example, in the equation, $2H_2 + O_2 \rightarrow 2H_2O$,
2 moles of hydrogen react with 1 mole of oxygen to form 2 moles of water
Also, 2 molecules of hydrogen react with 1 molecule of oxygen to form 2 molecules of water

Proportion Problem
Given the equation $2H_2 + O_2 \rightarrow 2H_2O$
(a) what is the ratio of moles of hydrogen to moles of oxygen?
(b) what is the ratio of moles of oxygen to moles of water?
(c) How many moles of water are produced from 8 moles of oxygen?

Valence (or Valency)

Memorize the entries in the following table:

Valence: Valence of an element or a radical is its combining capacity.

Some Common Cations and Anions with their Charges (Valences or valencies)

Monovalent**+1**	**Divalent: +2**	**Trivalent +3**
Li^{+1} (lithium) Na^{+1} (sodium) K^{+1} (potassium) NH_4^{+1} (ammonium) HC_3^{+1} Ag^{+1} (silver) Au^{+1} (gold) Hg^{+1}(mercury I; mercurous) Cu^{+1}(cuprous: copper I)	Mg^{2+} (magnesium) Ca^{2+} (calcium) Ba^{2+} (Barium) Zn^{2+} (zinc) Hg^{2+} (mercuric: mercury II Fe^{2+} (Ferrous: Iron II) Ni^{2+} (nickel II: nickelous)) Mn^{2+}(manganous:manganeseII) Cu^{2+} (cupric: copper II) Co^{2+} (cobaltous: cobalt I) Sn^{2+} (:tin II: stannous) Pb^{2+} (lead II; plumbous) Cr^{2+} (chromous; chromium II)	Al^{3+} (alumnum) Cr^{3+} (chromic; chromium III) Mn^{3+}(Manganic; manganese III) Fe^{3+} (ferric:Iron III) Co^{3+}(cobaltic: cobalt III) Ni^{3+}(Nickel III or nickelic)
-1	-**2**	**-3**
F^- (flouride) Cl^- (chloride) Br^- (bromide) I^- (iodide) CN^- (cyanide) HCO_3^-(bicarbonate) HSO_4^-(bisulfate) NO_3^- (nitrate) NO_2^- (nitrite) OH^- (hydroxide) ClO_4^- (perchlorate) ClO_3^- (chlorate) ClO_2^- (chlorite) ClO^- (hypochlorite) $H_2PO_4^-$(dihyrogen phosphate) MnO_4^-(permanganate) $C_2H_3O_2^-$(acetate)	S^{2-} (sulfide) O^{2-}(oxide) O_2^{2-}(peroxide) SO_4^{2-}(sulfate) SO_3^{2-}(sulfite) CO_3^{2-}(carbonate) $Cr_2O_7^{2-}$ (dichromate) CrO_4^{2-}(chromate) $S_2O_3^{2-}$ (thiosulfate) $C_2O_4^{2-}$ (oxalate)	PO_4^{3-}(phosphate) P^{3-} (phosphide) BO_3^{3-} (borate)

Extra: Valence , +4 Sn^{4+}(tin IV: stannic); Pb^{4+} (lead IV)

How to write chemical formulas using valences

Rule 1: We use the valences (disregarding the signs) as subscripts

Rule 2: The more electropositive element or radical (metal, metallic radical) is written first , followed by the non metallic (or the less electropositive element.

Rule 3: The valences of the electropositive and the electronegative elements or radicals are interchanged, but if the subscripts are the same, they are omitted.

Rule 4 : A radical should be enclosed in parenthesis if followed by a subscript other than 1.

Example 1 Write a formula for potassium chloride
Using K **(+1)** and Cl (-1), we obtain KCl

Example 2: Write a formula for Barium sulfate
Using Ba (+2) and SO_4(-2), we obtain $BaSO_4$

Example 3: Write a formula for Ferrous phosphate
Using Fe **(+2)** and PO_4(-3), we obtain $Fe_3(PO_4)_2$

Example 4: Write a formula for Ferric phosphate
Using Fe **(+3)** and PO_4(-3), we obtain $FePO_4$

To Master Contents

Mathematical Modeling

Some Reciprocal Relationships

1. Arithmetic If A working alone can do a piece of work in time t_A; B working alone can do the same work in time t_B; C working alone can do the same work in time t_C, and if A, B, and C working together, can do the same work in time t_{ABC}, then

$$\boxed{\frac{1}{t_{ABC}} = \frac{1}{t_A} + \frac{1}{t_B} + \frac{1}{t_C}}$$

That is, the reciprocal of the working-together time equals the sum of the reciprocals of working-alone times (individual times).

2. Geometry: For any triangle, the reciprocal of the inradius (R) equals the sum of the reciprocals of the exradii (r_1, r_2, and r_3).

Thus $\boxed{\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}$

3. Physics (Electricity) For electrical resistances in parallel (in an electric circuit), the reciprocal of the combined resistance, R, equals the sum of the reciprocals of the separate resistances, r_1, r_2, and r_3.

Thus $\boxed{\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}}$

4. Physics (Optics)

For two thin lenses in contact, the reciprocal of the combined focal length, F, equals the sum of the reciprocals of the separate focal lengths, f_1 and f_2, .

Thus $\boxed{\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2}}$

5. Physics (Optics) For spherical mirrors and thin lenses, the reciprocal of the focal length F equals the sum of the reciprocals of the object distance, d_o and the image distance d_i.

Thus $\boxed{\frac{1}{F} = \frac{1}{d_o} + \frac{1}{d_i}}$

6. Physics (Mechanics). If two bubbles of radii r_1, r_2, coalesce into a double bubble, the radius,

R, of the partition is given by

$$\boxed{\frac{1}{R} = \frac{1}{r_1} - \frac{1}{r_2}}$$

To Master Contents

TOPICS

Arithmetic

Integrated Arithmetic covers: Basic Definitions ;Terminology; and Types of Numbers; Writing Whole Numbers Using Numerals and Words; Basic Operations and Properties; Order of Operations and Evaluation of Arithmetic Expressions; Rounding-off Whole Numbers and Decimals; Estimation; Prime Numbers, Divisibility Rules; Prime Factorization; Least Common Multiple (LCM); Operations on Fractions and Mixed Numbers; Addition and Subtraction of Fractions; Comparison of Fractions and Subtraction of Mixed Numbers; Multiplication and Division: of Fractions and Mixed Numbers; Operations on Decimals; Comparison of Decimals ; Complex Decimals; Dividing Decimals; Converting Fractions to Decimals; Ratio and Proportion; Proportion Problems; Percent (%) and Calculations Involving Percent; Averages; Profit and Loss ; Areas and Perimeters; Bar, Line and Circle (Pie) Graphs; Scientific Notation; Measurements.

Elementary Algebra

Elementary Algebra covers: Signed Number and Real Number Operations; Order of Operations and Evaluation of Expressions; Exponential Notation and Rules of Exponents; Polynomial addition, subtraction, multiplication, and division; Solving First Degree Equations;Word Problems; Ratio and Proportion; Factoring Polynomials; Solving quadratic equations by factoring & applications; Graphs, Slopes, Intercepts and Equations of Straight Lines; Solving Systems of Linear Equations and Word Problems; Radicals, square roots, addition & multiplication of radicals; Pythagorean Theorem and Applications; Areas and Perimeters; Algebraic Fractions (reduction, multiplication, division & addition); Solving Linear inequalities.

Intermediate Algebra

Intermediate Algebra covers: Real Number Operations; Exponents ; Radicals; Fractional Exponents; Factoring Polynomials; Solving quadratic equations and applications; Graphs, Slopes, Intercepts, and Equations of Straight Lines; Graphs of Parabolas; Linear Inequalities; Compound Inequalities; Inequality Word Problems; Reduction, multiplication, division, and addition of algebraic fractions; Solving Fractional or Rational Equations; Solving Radical Equations; Variation and Variation Problems. Complex Numbers; Square roots of negative Numbers; addition, multiplication and division of complex Numbers; Absolute value equations; Absolute Value Inequalities; Logarithms; Logarithmic equations and Exponential Equations; Graphs of exponential and logarithmic functions; Applications of exponential and logarithmic functions. One-to-One Functions, Composite Functions
Inverse Functions and Inverse Relations

Intermediate Mathematics (US)

Intermediate Mathematics covers the following topics: Review of Operations; Exponents, Radicals. and operations on radical and Fractional Exponents; Reduction of Indices; Factoring Polynomials; Solving quadratic equations by and applications; Graphs, Slopes, Intercepts, and Equations of Straight Lines; Graphs of Parabolas; Linear Inequalities; Compound Inequalities; Inequality Word Problems; Reduction, multiplication, division, and addition of algebraic fractions; Solving Fractional or Rational Equations; Radical Equations; Complex Numbers; Absolute value equations; Absolute Value Inequalities; Logarithms; Logarithmic equations and Exponential Equations; Variation and Variation Problems; Basic Areas and Perimeters of triangles, rectangles, trapezoids, circles, and composite figures; Congruency Theorems; Similar Triangles; Right triangle trigonometry; Functional value of any angle; Laws of sines and cosines. Trigonometric Identities; Trigonometric equations.

To Master Contents

College Algebra

College Algebra covers: the following: Exponents; Radicals; Quadratic Equations; Polynomial Equations; Nonlinear Equations; Radical Equations; Sets, Relations, Functions; Excluded Values, Domain and Range; One-one Functions, Composite Functions; Inverse Functions; Linear Equations; Intercepts; Slopes; Perpendicular and Parallel Lines; Midpoint; Distance Formula; Variation; inequalities: Set Operations; Interval Notation ; Linear Inequalities; Compound Inequalities; Absolute Value Equations and Inequalities; Linear Inequalities in two Variables; Linear Programming; Quadratic Inequalities; Factored Inequalities; Graphs Involving Absolute Values; Greatest Integer Function; Step Function; Unit Step Function; Sign Function; Positive, Negative, Increasing, and Decreasing Functions ; Critical Points; Reflection of Points, Lines and Curves; Translation of Functions ; Contraction and Expansion of Curves; Symmetry; Even and Odd Functions: Graphs of Polynomial Functions; Rational Functions: Continuous and Discontinuous Functions; Asymptotes; Graphs of Rational Functions; Logarithms; Exponential and Logarithmic Equations; Graphs of Exponential and Logarithmic Functions; System of Linear Equations; Matrix and Matrix Methods; Determinants; Complex Numbers and Operations with Complex Numbers; Polar Form of Complex Numbers; Powers of Complex Numbers; De Moivre's Theorem; Roots of Complex Numbers ; Graphing Polar Coordinates and Equations; Conic Sections; Circles, Parabolas; Ellipses; Hyperbolas; Synthetic Division; Remainder and Factor Theorems; Descartes' Rules; Rational Roots;; Partial Fractions; Sequences and Series; Summation Notation; Arithmetic Sequence and Series; Geometric Sequence and Series; Factorial Notation; Binomial Theorem; Permutations and Combinations; Mathematical Induction.

College Trigonometry

College Trigonometry (40 Lessons) covers: review of functions; review of Geometry; Right Triangle Trigonometry; Angles of Elevation and Depression; Bearing; Linear Interpolation;Trigonometric Functional Value of any Angle; Functional Value given the Measure of the Angle; Trigonometric Functional Values of Quadrantal Angles; Finding Other Trigonometric Functional Values; Trigonometry of Oblique Triangles; Laws of Cosine and with Application s of Trigonometry to Vectors; Representation of Vectors; : Addition (Sum, Resultant, or Composition) of Vectors; Trigonometry of Real Numbers; Radian Measure; Arc Length; Reference Number ; Trigonometric Functional Values of Angles and of Real Number; Graphs of Trigonometric Functions; Periodicity of Trigonometric Functions; Graphs of $y = \sin x$, $y = \cos x$; $y = \tan x$; $y = \csc x$; $y = \sec x$; $y = \cot x$;; Sketching the Graphs of $y - k = a\sin(bx - h)$ and $y - k = a\cos(bx - h)$; Inverse Trigonometric Functions; Operations Involving Inverse Trigonometric Functions; Graphs of $y = \text{Arcsin}\, x$; $y = \text{Arccos}\, x$, and $y = \text{Arctan}\, x$; Trigonometric Identities; Proving Trigonometric Identities; Applications of the Sum and Difference Identities;   Solutions of Trigonometric Equations; and measurements

To Master Contents

Calculus 1 & 2

Calculus 1 & 2 covers differentiation and integration of functions using a guided and an analytical approach. All the normally difficult to understand topics have been made easy to understand, apply and remember. The topics include continuity, limits of functions; proofs; differentiation of functions; applications of differentiation to minima and maxima problems; rates of change, and related rates problems. Also covered are general simple substitution techniques of integration; integration by parts, trigonometric substitution techniques; application of integration to finding areas and volumes of solids. Guidelines for general approach to integration are presented to help the student save trial-and-error time on examinations. Other topics include L'Hopital's rule, improper integrals; differentiation and integration of hyperbolic functions; and memory devices to help the student memorize the basic differentiation and integration formulas, as well as trigonometric identities. This book is one of the most user-friendly calculus textbooks ever published

Power of Ratios covers the following::
definition and reduction of ratios to lowest terms; using ratios to compare quantities; using ratios to divide a quantity into parts; direct and inverse proportion; methods for solving direct proportion problems; methods for solving inverse proportion problems; compound proportion problems; geometric applications of ratios: similar triangles; theorems and proofs; comparison of congruency and similarity of triangles; applications of similarity theorems; radian-degree conversions; right triangle trigonometry and applications; straight lines: slopes of lines; intercepts and equations of straight lines; applications of ratios and proportion in physics and chemistry: Boyle's Law; Charles' Law; Gay-Lussac's Law; combined gas laws; dosage calculations in nursing; food preparation & nutrition; applications of ratios in engineering: machine design; model-prototype design; science and engineering ratios; applications of ratios in business; miscellaneous applications. Other topics include review of fractions; decimals; percent (%) and calculations involving percent; review of first degree equations containing one variable; axioms for solving equations; solving first degree equations. Other topics cover measurements; standard unit, error, and rounding-off numbers
The bonus topic covers solutions of 3-D Navier-Stokes equations of science and engineering by Power of Ratios;